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Valuation of Variance Swaps in Volatile Markets with Regime Switching

Mariam Zuwaid Salem Khamis AlShamsi

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United Arab Emirates University

College of Science

Department of Mathematical Sciences

VALUATION OF VARIANCE SWAPS IN VOLATILE MARKETS
WITH REGIME SWITCHING

Mariam Zuwaid Salem Khamis AlShamsi

This thesis is submitted in partial fulfillment of the requirements for the degree of Master
of Science in Mathematics

Under the Supervision of Dr. Youssef El Khatib

May 2020

Declaration of Original Work

I, Mariam Zuwaid Salem Khamis AlShamsi, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*Valuation of Variance Swaps in Volatile Markets with Regime Switching*", hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Youssef El Khatib, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Date 12/07/2020

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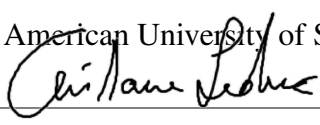
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Abstract

Stochastic differential equations (SDEs) are extensively used to model various financial quantities. In the last decades, financial modeling by SDEs under regime-switching have been utilized to allow moving from an economic state to another. The aim of this research work is to tackle the pricing of variance swaps in a volatile market under regime switching model. SDEs under regime-switching models are more realistic but the solution is more complicated and may not exist analytically. Therefore, numerical methods for finance are explored. The study proposes a new SDE under regime-switching with high volatility model for the prices of the underlying financial asset. The suggested model combines two existing models, the first one is on high volatile situations and the second is on regime-switching. Under these setting, the valuation of variance-swaps is investigated. As an application, a study of two states is developed: state A when the economy is going well and state B when the economy is under stress. Numerical techniques for finance are employed to obtain a solution for the pricing problem. Several illustrations of the solution are provided and show the efficiency of the used methods.

Keywords: Variance swaps, Regime switching, Brownian motion, Increased volatility, Markov chain.

Title and Abstract (in Arabic)

تقييم مقايضة التباين في الأسواق المتقلبة مع تبدل الحالة الاقتصادية

الملخص

تستخدم المعادلات التفاضلية العشوائية على نطاق واسع لنموذجة الكميات المالية المختلفة. في العقود الأخيرة، تم استخدام النموذجة المالية من خلال تحويل الأنظمة للسماح بالانتقال من حالة اقتصادية إلى أخرى. إن الهدف من هذا العمل البحثي هو معالجة تسعير المقايضات المتباينة في الأسواق المتغيرة وفق نموذج يراعي التغير في النظام الاقتصادي من حالة إلى أخرى. يعد استخدام المعادلات التفاضلية العشوائية أكثر واقعية من الناحية العلمية ولكن إيجاد حل مناسب لها يعد أمراً معقداً. في أغلب الأحيان لا يمكن إيجاد حل من الناحية التحليلية ولهذا يلجأ الباحثون إلى استعمال أدوات التحليل العددي. يقترح هذا العمل وضع معادلة تفاضلية جديدة لنموذج للأصول المالية الأساسية ذات مقياس تقلب للسعر عالي جداً. يغطي النموذج المقترح للحالات شديدة التقلب وكذلك نموذج يحاكي تغير الوضع الاقتصادي. تم دراسة تقييم مقايضة التباين للنموذج الجديد المعروض في هذا البحث. وافضت الدراسة إلى الحصول على نتائج نظرية. كما استخدمت التقنيات الرقمية للحصول على حل لمشكلة التسعير عندما لم يكن ممكناً الحصول على نتائج نظرية. كمثال، درست حالتين مختلفتين: عندما يكون الاقتصاد جيد وعندما يكون الاقتصاد سيئاً. لدعم نتائج هذا العمل تم أيضاً القيام ببرمجة النتائج بلغة السي بلاس بلاش للحصول على العديد من الرسوم التوضيحية للحل وإضافتها لهذا البحث. تظهر هذه الرسوم أهمية النموذج المقترح وكفاءة الطرق المستخدمة.

مفاهيم البحث الرئيسية: مقايضة التباين، تبدل الحالة الاقتصادية، نظام برونيان، أصول مالية شديدة التقلب، سلسلة ماركوف.

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Dedication

To my beloved family and teachers

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Chapter 1: Introduction

A financial derivative is a financial asset whose price alters on (i.e. is derived from) an underlying asset. It is an agreement between two dealers that is contingent on something happening in the future, for instance the future price of an underlying asset. Financial derivatives are commonly used to hedge against fluctuations in the underlying asset value. They represent one of the most important tools in risk-management. In general, a financial derivative could be constructed on a given financial product such as a commodity, a stock or a currency. Some derivatives products are dependent on the volatility or the variance of a given underlying asset. Among the most popular derivatives are options, futures (forwards) and swaps.

A derivative product does not have in general a well determined price since it is dependent on future events. Therefore, pricing of derivatives requires sophisticated mathematical tools particularly from probability theory and stochastic calculus. For decades, one of the most active research topics in financial mathematics literature is the pricing problem of financial derivatives.

The pricing of a derivative security consists essentially of dealing two problems which are prediction of asset price trajectory and derivative evaluation. The first problem deals with investigating a financial asset price advancement in time whilst the second one targets the calculation of an intrinsic value of the derivative security.

The purpose of this work is to study the pricing of a specific type of financial derivatives called variance swaps for a model that accounts for high volatile situations and under regime switching. There is a plenty of works on the valuation of variance swaps under regime switching in the literature. The main contribution of this thesis is dealing with the valuation of variance swaps under regime switching for markets under stress.

1.1 Content of the thesis

This thesis is structured as follows. After this introduction, an outline of the prevailing literature on the pricing of variance swaps is provided. The main contribution of this thesis is discussed in Chapter 3 and Chapter 4.

In the third chapter, a model for high volatility situations in the asset process will be defined. This is mainly the model suggested by [5] for European options but utilized in this chapter for evaluating variance swaps. This prediction model is a generalization of the Black and Scholes model [1] and is still advantageous since it leads to closed form solutions. Under this model, the price of a variance swap including the derivation of all the related formulas and proofs are obtained and given in this chapter. Though, as many other models in the existing literature, it does not account for various economic states.

In Chapter 4, an increased volatility model will be treated where the assumption on different economic states is permitted. More precisely, the main model in the thesis, the regime switching model, lets the interest rate and the volatility parameters dependent of a Continuous Time Markov Chain (CTMC). This model consequently contributes to the existing literature by studying the pricing problem for variance swaps where the underlying asset dynamic obeys a regime switching model with abnormal volatility. Numerical simulations for the price of a variance swap under this model are provided. Several illustrations for this price are performed. Comparisons with the model with an augmented volatility with no regime switching and also with the model with regime switching but without increase in the volatility are presented.

Chapter 2: Variance Swaps

A swap is an agreement to exchange cash flows in the future according to prearranged formula. A basic swap contract consists of exchanging cash flows calculated using one floating reference rate against cash flows calculated using another floating reference rate. There exist various types of swaps. Some of them are listed below:

- Swaps where the notional principle is an increasing function of time are known as step-up swaps. If the notional principle is a decreasing function of time then it is known as amortizing swaps.
- A currency swap involves exchanging principle and interest payments in one currency for principle and interest payments in another.
- A differential swap, sometimes referred to as a diff swap, is an interest rate swap where a floating interest rate is observed in one currency and applied to a principle in another currency.
- In an equity swap, one party promises to pay the return on an equity index on a notional principle, while the other promises to pay a fixed or floating return on a notional principle.
- A cancelable swap is a plain vanilla interest rate swap where one side has the option to terminate on one or more payment dates.[7]

2.1 Introduction to volatility and variance swaps

2.1.1 Definition

Volatility is a measure of the variations in the price of a financial product. There are two different types of volatility: implied and realized. The present market price for volatility is the implied volatility, which acts like the fair volatility price built on the mar-

ketplace's expectation for movement over a stated period of time. On the other hand, realized volatility is computed from the variations in the underlying price over a given period. The realized volatility is known as historical volatility when the given period is in the past and if the period is in the future it is called future realized volatility.

Volatility swaps are forward agreements on the future realized volatility of the returns of the selected underlying asset. Variance swaps are forwards contracts analogous to volatility swaps, but based on realized variance, the square of the future volatility. These derivatives give investors the opportunity to trade volatility just as they wish to trade an underlying financial asset. They allow to exchange future realized volatility against current implied volatility. However, volatility and variance swaps are not conventional swaps that involve a simple exchange of cash flows between counter-parties. In fact, at settlement, the payoff for the long position of a volatility swap is equal to the annualized realized volatility over a pre-specified period minus the volatility strike of the contract multiplied by a notional amount of the swap in dollars per annualized volatility point. At maturity, the payoff is Notional Amount \times (Volatility – Volatility Strike). The short position is just the opposite.

2.1.2 Pricing

Now, consider a variance swap on an agreed underlying market asset S observable during the term of the swap agreement with price $(S_t)_{t \in [0, T]}$. The volatility of S is denoted by $(\sigma_t)_{t \in [0, T]}$. Let N represent the notional amount of the variance swap in dollars per annualized volatility point squared. Moreover, let σ_R be the realized volatility (in annual terms) of the underlying asset S . The realized variance defining the pay-off is given by:

$$\sigma_R^2 = \frac{1}{T} \int_0^T \sigma_t^2 dt. \quad (2.1)$$

If the the variance strike is denoted by K_{Var} then the payoff at maturity T is given by:

$$N(\sigma_R^2 - K_{Var}). \quad (2.2)$$

There exist two definitions of the historical volatility depending on how the returns have been calculated. This can be utilized by either log-returns expressed by:

$$\frac{1}{n} \sum_{i=1}^n \log \left(\frac{S_i}{S_{i-1}} \right). \quad (2.3)$$

Or arithmetic returns given by:

$$\frac{1}{n} \sum_{i=1}^n \frac{S_i - S_{i-1}}{S_{i-1}}, \quad (2.4)$$

where S_i are quoted prices at interval times indexed by i . If the S_i are taken as daily closing prices then the variance needs to be converted into a per annum basis (in terms of annualized variance).

Remark: The formula in (2.1) provides a definition for a continuous time realized variance, when it is utilized it means continuous variance swaps. As mentioned in [8] continuous variance swaps prices can be considered as approximations to the actual values. This is because practitioners are commonly replacing the integral in the Equation (2.1) by a discrete sum. They actually compute the realized variance from closing prices using one of the the formulas (2.3) or (2.4) to calculate the returns of the underlying asset price, which are utilized to compute discrete annualized variance. In such a case variance swap is known as a "discretely-sampled" variance swap. In this thesis, discretely-sampled variance swap with arithmetic returns will be treated where the realized annualized volatility are computed based on returns that are given by (2.4).

Example 2.1.1. As an example of a variance swap contract is a notional amount of $N = 100,000$ dollars per (one volatility point)², with delivery swap rate of $K_{Var} = (15\%)^2$ per

annum on the *S&P500* daily closing index and maturity of 1 year.

Let the time interval be discretized into M equal sub-intervals with M monitoring dates $0 = t_0 < t_1 < \dots < t_M = T$. Let AF be the annualization factor. For example $AF = 12$ for monthly monitoring, $AF = 52$ for weekly monitoring, or $AF = 252$ for daily monitoring. If the log-return (2.3) is used, then the discrete annualized realized variance over $[0, T]$ is defined by:

$$V_M := \frac{AF}{M} \sum_{m=1}^M \left(\log \frac{S_{t_m}}{S_{t_{m-1}}} \right)^2 * 100^2. \quad (2.5)$$

However, in the case of arithmetic returns (2.4) the discrete annualized realized variance over $[0, T]$ is defined by:

$$V_M = \frac{AF}{M} \sum_{m=1}^M \left(\frac{S_{t_m} - S_{t_{m-1}}}{S_{t_{m-1}}} \right)^2 * 100^2. \quad (2.6)$$

Here the time step size is $\Delta t = \frac{T}{M}$, then $AF = \frac{1}{\Delta t}$ so that $\frac{AF}{M} = \frac{1}{T}$. As mentioned before, the aim is to find the variance swap price using the arithmetic return formula.

Remark: 1. A discretely-sampled variance swap is a forward contract that exchanges the discrete realized variance with a fixed strike. To determine the fair strike or the fixed swap rate K_{Var} of the variance swap, the price of the contract is assumed to be 0 at inception. It's value is given by the risk neutral expectation of V_M , i.e,

$$K_{Var} = E^Q[V_M]. \quad (2.7)$$

2. A volatility swap is a contract which is similar to the discrete variance swap. It's fair strike is given by:

$$K_{Vol} = E^Q[\sqrt{V_M}]. \quad (2.8)$$

3. There are options built on discrete realized variance, an option offers to the owner the

right but not the obligation to exchange the realized variance for a fixed strike. As an example, the price of an European call option with strike K is given by:

$$C = E^Q[e^{-rT}(V_M - K)^+]. \quad (2.9)$$

In this thesis the main goal is to find the value of a discretely-sampled variance swap as given in Equation (2.7) where V_M is obtained from arithmetic return as expressed in (2.6). In Chapter 3 the price of a discretely-sampled variance swap will be derived using a high volatility model. In Chapter 4, numerical methods are employed to obtain the variance swap price for a regime switching high volatile model. In the next section, some tools from stochastic calculus needed for the study are provided. More detailed concepts and tools from probability theory and elementary stochastic calculus are presented in the appendices.

2.2 Preliminaries from stochastic calculus

In this section, some stochastic calculus tools and concepts required in the study will be presented such as stochastic processes, Brownian motion, Martingales, stochastic integration and others notions from probability theory are presented in the appendices.

2.2.1 Stochastic processes and Brownian motion

Let I be a subset of the interval $[0, \infty)$. A family of random variables $(X_t)_{t \in I}$, indexed by I , is called a stochastic (or random) process, when $I = N$, and $(X_t)_{t \in I}$, is said to be a discrete-time process, and when $I = [0, \infty)$, it is called a continuous-time process. Another definition for a stochastic process can be as follows: A stochastic process is a parameterized collection of random variables $(X_t)_{t \in I}$, defined on a probability space (Ω, F, P) , and assuming values in \mathbb{R}^n . Consider the probability space (Ω, F, P) . A filtration on (Ω, F, P) is an increasing family $(F_t)_{t \geq 0}$ of sub σ -algebras of F . In other words, for each t , F_t is a σ -algebra included in F and if $s \leq t$, $F_s \subset F_t$. A probability space (Ω, F, P) endowed with a filtration $(F_t)_{t \geq 0}$ is called a filtered probability space $(\Omega, F, (F_t)_{t_0 \leq t \leq T}, P)$.

Definition 2.2.1. Let (F_t) be a filtration on (Ω, F, P) . A stochastic process (X_t) is said to be F_t -adapted if $\forall t \geq 0$, X_t is F_t -measurable.

A Brownian motion is a Wiener stochastic process. A Wiener process is a stochastic process $W(t)$ with values in \mathbb{R} defined for $t \in [0, \infty)$ such that it satisfies the following four conditions:

1. $W(0) = 0$.
2. If $0 < s < t$, then $W(t) - W(s)$ has a normal distribution $N \sim (0, t - s)$ with mean 0 and variance $(t - s)$.
3. If $0 \leq s \leq t \leq u \leq v$ (i.e., the two intervals $[s, t]$ and $[u, v]$ do not overlap) then $W(t) - W(s)$ and $W(v) - W(u)$ are independent random variables.
4. The sample paths $t \rightarrow W(t)$ are almost surely continuous.

2.2.2 Martingales and Markov property

Definition 2.2.2 (Martingale). Let $(F_t)_{t \geq 0}$ be a filtration on (Ω, F, P) . A stochastic process $(M_t)_{t \geq 0}$ is called a martingale if it is satisfying the following properties:

1. (M_t) is F_t -adapted.
2. $E||M_t|| < \infty, \quad \forall t \geq 0$.
3. $E(M_t/F_s) = M_s, \quad \forall 0 \leq s \leq t$.

Remark: (i) If the condition (3) of the previous definition is replaced by $E(M_t/F_s) \geq M_s, \quad \forall \quad 0 \leq s \leq t$, then (M_t) is called submartingale.

(ii) If the condition (3) of the previous definition is replaced by $E(M_t/F_s) \leq M_s, \quad \forall \quad 0 \leq s \leq t$, then (M_t) is called supermartingale.

(iii) A positive submartingale is a submartingale $(X_t)_{t \geq 0}$ satisfying $X_t \geq 0 \quad \forall \quad t \geq 0$.

Proposition 2.2.1 (Tower Property). Consider $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ and a random variable X .

Then

$$E[E[X|\mathcal{G}_2]|\mathcal{G}_1] = E[E[X|\mathcal{G}_1]|\mathcal{G}_2] = E[X|\mathcal{G}_1]. \quad (2.10)$$

Theorem 2.2.1 (Martingale Representation Theorem). Let $W := (W_t)_{t \in [0, T]}$ be a Brownian motion on (Ω, \mathcal{F}, P) . Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by W . Let $(X_t)_{t \in [0, T]}$ be a martingale, then there is an adapted process $(\xi_t)_{t \in [0, T]}$ such that:

$$X_t = X_0 + \int_0^t \xi_u dW_u, \quad 0 \leq t \leq T. \quad (2.11)$$

Definition 2.2.3 (Markov property). An adapted stochastic process $(X_t)_{t \geq 0}$ has the Markov property if for every bounded \mathcal{B} -measurable function ϕ the following is hold:

$$E[\phi(X_T)|\mathcal{F}_t] = E[\phi(X_T)|X_t]. \quad (2.12)$$

2.2.3 Stochastic integration

Definition 2.2.4. $M^p([0, T], \mathbb{R})$ is defined to be the subspace of $L^p([0, T], \mathbb{R})$ such that for any process $(X_t) \in^p([0, T], \mathbb{R})$ the following is correct:

$$E\left(\int_0^T |X(t)|^p dt\right) < \infty, \quad (2.13)$$

where $L^p([0, T], \mathbb{R})$ is defined as $E\left(\int_0^T |X(t)|^p dt\right)^{\frac{1}{p}}$.

Consider a Brownian motion W and a stochastic process (X_t) both adapted to a

given filtration (F_t) . The stochastic integral is defined by the following expression:

$$I_t(X) = \int_0^t X(s) dW(s). \quad (2.14)$$

Now, the stochastic integral of a simple process will be presented and then some of its properties will be defined.

Definition 2.2.5. (Elementary process or simple process). A process $(X_t)_{t \in \mathbb{R}} \in L^P([0, T], \mathbb{R})$ is called simple or elementary process if there exist a partition $0 = t_0 < t_1 < \dots < t_n = T$ such that:

$$X_s(\omega) = \sum_{j=0}^n 1_{[t_j, t_{j+1}]} \theta_j(\omega), \quad (2.15)$$

where θ_j is a bounded F_{t_j} -measurable random variable.

Definition 2.2.6. (Itô's integral). The Itô's Integral of the simple process $(X_t)_{t \in \mathbb{R}} \in L^2([0, T], \mathbb{R})$ is defined by:

$$I_t(X) = \int_0^t X(s) dW(s) := \sum_{j=0}^{n-1} \theta_j (W_{t_{j+1}} - W_{t_j}). \quad (2.16)$$

Lemma 2.2.1 If f is an elementary function in $L^2([a, b], \mathbb{R})$ and W_t is a Brownian motion, then

$$1. \ E \left(\int_a^b f(t) dW_t \right) = 0. \quad (2.17)$$

$$2. \ E \left(\int_a^b f(t) dW_t \right)^2 = \int_a^b E(f^2(t)) dt \text{ (Itô Isometry)}. \quad (2.18)$$

Proof. 1. The definition gives:

$$\int_a^b f(t) dW(t) = \sum_{j=0}^{n-1} f_j (W_{t_{j+1}} - W_{t_j}).$$

By taking expectation in both sides:

$$E \left[\int_a^b f(t) dW(t) \right] = \sum_{j=0}^{n-1} E(f_j) E(W_{t_{j+1}} - W_{t_j}) = 0,$$

since $W_{t_{j+1}} - W_{t_j}$ is a normal distribution with mean 0 and standard deviation

$$\sqrt{t_{j+1} - t_j}.$$

2.

$$\begin{aligned} \left(\int_a^b f(t) dW(t) \right)^2 &= \left[\sum_{j=0}^{n-1} f_j (W_{t_{j+1}} - W_{t_j}) \right]^2 \\ &= \sum_{j=0}^{n-1} (f_j)^2 (W_{t_{j+1}} - W_{t_j})^2 \\ &\quad + \sum_{l=0}^{n-1} \sum_{k=0, k \neq l}^{n-1} f_l f_k (W_{t_{l+1}} - W_{t_l})(W_{t_{k+1}} - W_{t_k}). \end{aligned}$$

Taking expectation in both sides and using independence of the increments of Brownian motion, this gives:

$$\begin{aligned} E \left(\int_a^b f(t) dW(t) \right)^2 &= \sum_{j=0}^{n-1} E(f_j)^2 E(W_{t_{j+1}} - W_{t_j})^2 \\ &= \sum_{j=0}^{n-1} E(f_j)^2 E(t_{j+1} - t_j) \\ &= \int_a^b E(f^2(t)) dt. \end{aligned}$$

□

Proposition 2.2.2. For any process $X = (X_t)_{t \geq 0} \in M^2([0, T], \mathbb{R})$, such that $E|X_t|^2 < \infty$, $\forall t \geq 0$, there exist a sequence $(f_t^{(n)})_{t \geq 0}$ of simple process such that $E|f_t^{(n)}|^2 < \infty$ and $\lim_{n \rightarrow \infty} E \left[\int_0^t |X_s - f_s^{(n)}|^2 ds \right] = 0$.

Definition 2.2.7. For any process $X = (X_t)_{t \geq 0} \in M^2([0, T], \mathbb{R})$, a stochastic integral of X

with respect to a Brownian motion W is defined by:

$$\int_0^t X_s dW(s) := \lim_{n \rightarrow \infty} \int_0^t f_s^{(n)} dW(s), \quad (2.19)$$

where $f_s^{(n)}$ is the sequence of simple process converging to X using L^2 limit according to the previous proposition. Moreover, using Itô isometry for elementaries functions one can prove that the limit on this definition does not depend on the actual choice of $(f^{(n)})$.

Proposition 2.2.3. (*Properties of Itô integral*)

For any process $X = (X_t)_{t \geq 0} \in M^2([0, T], \mathbb{R})$, such that $E|X_t|^2 < \infty$, for any functions $f, g \in M^2([0, T], \mathbb{R})$ and $0 \leq S < U < T$, the following holds:

1. $\int_S^T f dW(t) = \int_S^U f dW(t) + \int_U^T f dW(t)$ almost surely.
2. $\int_S^T (cf + g) dW(t) = c \int_S^T f dW(t) + \int_S^T g dW(t)$, for any constant c .
3. $\int_S^T f dW(t)$ is F_T -measurable.
4. $E \left(\int_0^t X_s dW(s) \right) = 0$.
5. $E \left(\int_0^t X_s dW(s) \right)^2 = \int_0^t E(X_s^2) ds$.

The proof can be found in [10].

Proposition 2.2.4. For any elementary function $f^{(n)}$ F_t -adapted, the integral

$$I_n(t, \omega) = \int_0^t f^{(n)} dW(r) \quad (2.20)$$

is a martingale with respect to F_t .

See Oksendal [10] for more details.

Proof. For any $t \leq s$:

$$\begin{aligned}
E[I_n(s, \omega)/F_t] &= E \left[\left(\int_0^s f^{(n)} dW(r) \right) / F_t \right] \\
&= E \left[\left(\int_0^t f^{(n)} dW(r) \right) / F_t \right] + E \left[\left(\int_t^s f^{(n)} dW(r) \right) / F_t \right] \\
&= \int_0^t f^{(n)} dW(r) + E \left[\sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} f^{(n)} \Delta W_j / F_t \right] \\
&= \int_0^t f^{(n)} dW(r) + \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} E[f_j^{(n)} \Delta W_j / F_t] \\
&= \int_0^t f^{(n)} dW(r) + \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} E[E[f_j^{(n)} \Delta W_j / F_t] / F_t] \\
&= \int_0^t f^{(n)} dW(r) + \sum_{t \leq t_j^{(n)} \leq t_{j+1}^{(n)} \leq s} E[f_j^{(n)} E[\Delta W_j / F_t] / F_t] \\
&= \int_0^t f^{(n)} dW(r), \text{ since } E[\Delta W_j / F_t] = E[\Delta W_j] = 0 \\
&= I_n(t, \omega).
\end{aligned}$$

□

Proposition 2.2.5. (*Generalization*)

Let $f(t, \omega) \in M^2([0, T], \mathbb{R})$ for all t . Then the integral

$$M_t(\omega) = \int_0^t f(s, \omega) dW(s) \quad (2.21)$$

is a martingale with respect to F_t and

$$P \left[\sup_{0 \leq t \leq T} |M_t| \geq \lambda \right] \leq \frac{1}{\lambda^2} E \left[\int_0^T f^2(s, \omega) ds \right], \forall \lambda > 0. \quad (2.22)$$

The proof is in [10].

2.2.4 Stochastic Differential Equation (SDE) and Itô formula

Definition 2.2.8. (1-dimensional Itô process)

Let W_t be a 1-dimensional Brownian motion on (Ω, F, P) . An Itô process (or Stochastic integral) is any stochastic process X_t of the form

$$X_t = X_0 + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) dW(s), \quad (2.23)$$

where $u \in L^1([0, T], \mathbb{R})$ and $v \in L^2([0, T], \mathbb{R})$.

Proposition 2.2.6. (first 1- dimensional Itô formula)

Let (Ω, F, P) be a complete probability space, $(W_t)_{t \in \mathbb{R}_+}$ a one-dimensional Brownian motion and $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is twice derivable. If (X_t) is any process of the form (2.23), then $f(X_t)$ is an Itô processes and

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) u_s ds + \frac{1}{2} \int_0^t f''(X_s) v_s^2 ds + \int_0^t f'(X_s) v_s dW_s. \quad (2.24)$$

You can read the proof in [11].

One of the important notation for the stochastic integral define over a Brownian motion is that it follows the normal distribution. It means that:

$$\int_0^t f(s) dB_s \sim N \left(0, \int_0^t |f(s)|^2 ds \right). \quad (2.25)$$

Now, using the property of independent increments for Brownian motion, its second

moment is given by

$$\begin{aligned}
E \left[\int_0^t f(s) dB_s \right]^2 &= \sum_{i,j=1}^n f_{i1} f_{j-1} E[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \\
&= \sum_{j=1}^n |f_{j-1}|^2 E[(B_{t_j} - B_{t_{j-1}})^2] \\
&= \sum_{j=1}^n |f_{j-1}|^2 (t_j - t_{j-1})
\end{aligned}$$

This gives

$$E \left| \int_0^t f(s) dB_s \right|^2 = \sum_{j=1}^n |f_{j-1}|^2 (t_j - t_{j-1}) = \int_0^t |f(s)|^2 ds. \quad (2.26)$$

Now, the stochastic differential equation (SDE) will be presented. This is a new type of differential equations differ from the normal one. Talking about stochastic differential, means talking about a random process that evolve over time. For instance,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \text{which is the geometric Brownian motion,}$$

$$dY_t = \kappa(\theta - Y_t)dt + \gamma dW_t, \quad \text{which is an Ornstein-Uhlenbeck process,}$$

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad \text{which is a diffusion process.}$$

For more SDEs, refer to Papanicolaou [12].

However, the Itô formula is useful in many areas. Actually, there are two main areas. The first one is to find the solution of the SDEs and the second one is to price the financial derivatives products. Consider (2.26) and let $f : [a, b] \rightarrow \mathbb{R}$ be a twice continuously differentiable function ($f \in C^2[a, b]$). Then the process $f(X)$ is a continuous and

$$f(X_t) - f(X_0) = \int_0^t f'(X_t) dW_t + \frac{1}{2} \int_0^t f''(X_t) d\langle W \rangle_t. \quad (2.27)$$

and the formula with two variables $f(t, X_t)$ is given by:

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t} dt + \int_0^t \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2} d\langle X, X \rangle_t. \quad (2.28)$$

Note that, to determine the quadratic variation $d\langle W \rangle$, the following properties will be used:

$$\langle dW_t, dW_t \rangle = dt$$

$$\langle dW_t, dt \rangle = 0$$

$$\langle dt, dW_t \rangle = 0$$

$$\langle dt, dt \rangle = 0$$

Chapter 3: Variance Swap in High Discretely-Sampled Volatility

The aim of this chapter is to address the issue of pricing discretely-sampled variance swaps under high volatile models. First a review on how to find K_{var} in the Heston stochastic volatility model for a discretely-sampled model will be presented. The method is provided in section 1 and can be found in [15] and [16]. The main contribution in this chapter is given in section 2 where the value of discretely-sampled variance swap in high volatile model will derive partially. In the third section, numerical methods are performed and figures comparing the high volatility and the Heston model are provided.

From now on, and throughout Chapter 3 and Chapter 4 the following assumptions and notations are used (unless otherwise stated). All the work will be based on the following notations: a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathcal{F}_T, Q)$, a standard Wiener process $(W_t)_{t \in [0, T]}$ with $\mathcal{F}_t := \sigma(W_t)$ for any $t \in [0, T]$ and assume that Q is risk-neutral probability, which is known. The main goal is to find the value of a discrete variance swap. The underlying asset price is denoted by $(S_t)_{t \in [0, T]}$, the notional amount of the variance swap in dollars per annualized volatility point squared is L . Moreover, let σ_R be the realized volatility (in annual terms) of the underlying asset S computed using arithmetic return. The formula for the realized variance defining the pay-off is then given by

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=1}^N \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 * 100^2. \quad (3.1)$$

Assuming that there are total of N closing prices S_{t_i} of the underlying asset observed at an equally-spaced time t_i . In this case, and as mentioned in Chapter 2, a multiplication by $AF = \frac{1}{\Delta t} = \frac{N}{T}$ is needed to get the annualized variance in the above formula. The value of AF depends on the sampling frequency, for example, $AF = 12$, if the closing prices are

taken every month. Let the strike be K_{var} . At time t the value of variance swap is

$$V_t = e^{-r(T-t)} E^Q [L(\sigma_R^2 - k_{var}) | \mathcal{F}_t] = e^{-r(T-t)} E_t^Q [L(\sigma_R^2 - k_{var})], \quad (3.2)$$

where $E_t^Q = E^Q[\cdot | \mathcal{F}_t]$ is the conditional expectation at time t . Since at inception $V_0 = 0$ then by (2.7) and (3.1), the following is hold:

$$k_{var} = \frac{AF}{N} \sum_{i=1}^N E_0^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] * 100^2. \quad (3.3)$$

The problem of pricing variance swap consists in finding the fair variance delivery price k_{var} . The goal is then to evaluate the conditional expectations in (3.3). In other words, all the conditional expectations should be computed

$$E_0^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right], \quad (3.4)$$

for $i = 1, \dots, N$, which depends essentially on the SDE employed to predict the future values of the underlying asset S_{t_i} for $i = 1, \dots, N$. The next section summarizes the results on Heston model.

3.1 Discretely-sampled Variance swaps for Heston model

The Heston model [6] assume that the underlying asset price $(S_t)_{t \in [0, T]}$ is governed by the SDE

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t \\ dv_t = \kappa(\theta - v_t) dt + \sigma_v \sqrt{v_t} dW_t^v. \end{cases} \quad (3.5)$$

Here $(W_t^v)_{t \in [0, T]}$ is a Brownian motion independent from W_t and in this section the probability space $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathcal{F}_T, Q)$ is assumed to has filtration $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^{W^v}$ for any $t \in [0, T]$, where \mathcal{F}_t^W and $\mathcal{F}_t^{W^v}$ are the natural filtration generated by W and W^v

respectively. The risk-neutral probability Q is also assumed to be known. The constant κ, θ and σ_v are the risk-neutral parameters.

The approach to compute the above expectations in the case of Heston model as described in [15] consists first in rewriting the expectations in (3.4) as follows

Corollary 3.1.1. *The following is true:*

$$E_0^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] = 1 + E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} E_{t_{i-1}}^Q [S_{t_i}^2] \right] - 2E_0^Q \left[\frac{1}{S_{t_{i-1}}} E_{t_{i-1}}^Q [S_{t_i}] \right]. \quad (3.6)$$

Proof. Using the fact that i and then both t_i and t_{i-1} are considered as known constants and since $\mathcal{F}_0 \subset \mathcal{F}_{t_{i-1}}$ applying the tower property to (3.4) gives

$$\begin{aligned} E_0^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] &= E_0^Q \left[E_{t_{i-1}}^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] \right] \\ &= E_0^Q \left[E_{t_{i-1}}^Q \left[\frac{S_{t_i}^2}{S_{t_{i-1}}^2} \right] - 2E_{t_{i-1}}^Q \left[\frac{S_{t_{i-1}} S_{t_i}}{S_{t_{i-1}}^2} \right] + E_{t_{i-1}}^Q \left[\frac{S_{t_{i-1}}^2}{S_{t_{i-1}}^2} \right] \right]. \end{aligned}$$

Since $S_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$ -measurable then

$$\begin{aligned} E_0^Q \left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right] &= E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} (E_{t_{i-1}}^Q [S_{t_i}^2] - 2S_{t_{i-1}} E_{t_{i-1}}^Q [S_{t_i}]) + 1 \right] \\ &= E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} (E_{t_{i-1}}^Q [S_{t_i}^2]) \right] - 2E_0^Q \left[\frac{1}{S_{t_{i-1}}} E_{t_{i-1}}^Q [S_{t_i}] \right] + 1. \end{aligned}$$

□

Now the key idea of [15], to compute (3.4), utilizes Corollary (3.1.1) and can be summarized in two steps as follows

- Step 1: calculate the expectations $E_{t_{i-1}}^Q [S_{t_i}]$ and $E_{t_{i-1}}^Q [S_{t_i}^2]$.
- Step 2: plug the values of expectations obtained in step 1 into (3.6) then evaluate

the outcome expectation in the form $E_0^Q[\cdot]$.

Step 1 can be done by computing the conditional expectation of $Y_t := S_t^\gamma E_{t_{i-1}}^Q[Y_t] = E_{t_{i-1}}^Q[Y_t | (Y_{t_{i-1}} = y, v_{t_{i-1}} = v)]$, for all $t \in [t_{i-1}, t_i]$, where γ can be any non-zero real number especially 1 and 2 (cf. proposition 2.1 in [15]). The second step is accomplished in Proposition 2.2 in [16].

The main results of this chapter are presented in the next section. The techniques follow essentially the two steps explained before.

3.2 Variance swaps in high volatile model

There exist many articles for dealing with pricing variance swaps with different models for predicting the underlying asset price trajectory S_t . The reader can refer to [2], [4], [9] and [8], [16], [17] and [15] for more details about the suggested models and methods in the literature. The list of reference is not exhaustive. To the best knowledge, pricing variance swap under a high volatility model has previously never been addressed.

To examine the impact of a high volatility on the value of fair variance delivery strike K_{var} , the variance swap pricing problem when the underlying asset price has an augmented volatility is considered.

3.2.1 The high volatile model

More precisely, in this section, assume that $(S_t)_{t \in [0, T]}$ is given by the SDE

$$dS_t = rS_t dt + (\sigma S_t + \beta e^{rt}) dW_t, \quad S_0 > 0. \quad (3.7)$$

This model presents some practical advantages such as accounting for crisis situations where the prices are suffering from unusual and sudden depreciation. Moreover there exists a closed form solution for pricing European option in the case of this model. An additional advantage of using this model is that it is a stochastic volatility model that satisfies the leverage effect where volatility and asset price are inversely proportional. The

SDE (3.7) has the solution

$$S_t = (S_0 + \frac{\beta}{\sigma})e^{(r - \frac{\sigma^2}{2})t + \sigma W_t} - \frac{e^{rt}}{\sigma}. \quad (3.8)$$

See [5] for more details on the above solution and derivation of prices for European options.

3.2.2 Valuation of discretely-sampled variance swap

This subsection deals with the pricing of variance swap problem under the high volatility model (3.7) by investigating the valuation of strike K_{var} given by (3.3) which is reduced to the calculation of the conditional expectations (3.4). The way to do this is inspired from the method of [15] employed for Heston model stated in corollary 3.1.1 and steps 1 and 2 in the previous section. First apply Itô formula to have the SDE of stochastic processes power of S_t as provided in the next corollary.

Corollary 3.2.1. *Let γ be a non-zero real number and let $(Y_t)_{t \in [0, T]}$ be the process defined by $Y_t = S_t^\gamma$. Therefore*

$$dY_t = \gamma Y_t \left[\left(r + \frac{\gamma - 1}{2} (\sigma + e^{rt} Y_t^{-\frac{1}{\gamma}})^2 \right) dt + \left(\sigma + e^{rt} Y_t^{-\frac{1}{\gamma}} \right) dW_t \right]. \quad (3.9)$$

Proof. Applying Itô formula with the function $f(x) = x^\gamma$ and S_t given by (3.7). Then

$$\begin{aligned} dY_t &= d(S_t^\gamma) = d(f(S_t)) = f'(S_t) dS_t + \frac{1}{2} f''(S_t) d\langle S_t, S_t \rangle \\ &= \gamma(S_t)^{\gamma-1} [r S_t dt + (\sigma S_t + e^{rt}) dW_t] + \frac{1}{2} \gamma(\gamma-1) (S_t)^{\gamma-2} [(\sigma S_t + e^{rt})^2 dt]. \end{aligned}$$

Since

$$\begin{aligned}\langle dS_t, dS_t \rangle &= \langle rS_t dt + (\sigma S_t + e^{rt}) dW_t, rS_t dt + (\sigma S_t + e^{rt}) dW_t \rangle \\ &= (\sigma S_t + e^{rt})^2 dt.\end{aligned}$$

Therefore

$$\begin{aligned}dY_t &= d(S_t^\gamma) = \gamma r S_t^\gamma dt + \gamma (S_t)^{\gamma-1} (\sigma S_t + e^{rt}) dW_t + \frac{1}{2} \gamma (\gamma-1) (S_t)^{\gamma-2} (\sigma S_t + e^{rt})^2 dt \\ &= \left[\left(r + \frac{1}{2} (\gamma-1) S_t^{-2} (\sigma S_t + e^{rt})^2 \right) \gamma S_t^\gamma \right] dt + [\gamma (S_t)^{\gamma-1} (\sigma S_t + e^{rt})] dW_t. \\ &= \left[\left(r + \frac{1}{2} (\gamma-1) \left(Y_t^{\frac{1}{\gamma}} \right)^{-2} (\sigma Y_t^{\frac{1}{\gamma}} + e^{rt})^2 \right) \gamma \left(Y_t^{\frac{1}{\gamma}} \right)^\gamma \right] dt \\ &\quad + \left[\gamma \left(Y_t^{\frac{1}{\gamma}} \right)^{\gamma-1} (\sigma Y_t^{\frac{1}{\gamma}} + e^{rt}) \right] dW_t.\end{aligned}$$

The last equation can be simplified to get (3.9). This ends the proof. \square

In the next proposition, the conditional expectations $E_{t_{i-1}}^Q[Y_t]$ where $t \in [t_{i-1}, t_i]$ will be shown as a solution of a PDE-Partial Differential Equation.

Proposition 3.2.1. Let $t \in [t_{i-1}, t_i]$, then there exists a function $U_i^\gamma \in C^{1,2}([t_{i-1}, t_i] \times]0, \infty[)$ such that $E_{t_{i-1}}[Y_t] = U_i^\gamma(t, Y_t)$. Moreover $U_i^\gamma(t, y)$ is solution of the following PDE

$$\begin{cases} \partial_t U_i^\gamma + \gamma y \partial_y U_i^\gamma \left(r + \frac{\gamma-1}{2} (\sigma + e^{rt} y^{-\frac{1}{\gamma}})^2 \right) + \frac{1}{2} \gamma^2 y^2 (\sigma + e^{rt} y^{-\frac{1}{\gamma}})^2 \partial_{yy} U_i^\gamma = 0, \\ U_i^\gamma(t_i, y) = y. \end{cases} \quad (3.10)$$

Proof. Using the Markov property of $(Y_t)_{t \in [0, T]}$, then for any $s \in [0, t]$,

$$E_s[f(Y_t)] = E[f(Y_t) | Y_s = y] = U^\gamma(s, y), \text{ with } U \in C^{1,2}([0, t] \times]0, \infty[).$$

Consider the function:

$$U_i^\gamma : [t_{i-1}, t_i] \times]0, \infty[\longrightarrow \mathbb{R}$$

$$U_i^\gamma(t, y) = E[Y_t | Y_t = y].$$

So: $U_i^\gamma(t_{i-1}, y) = E[Y_{t_i} | Y_{t_{i-1}} = y]$ and $U_i^\gamma(t_i, y) = E[Y_{t_i} | Y_{t_i} = y] = E[y] = y$. Applying Itô formula to $U_i^\gamma(t, Y_t)$. It gives:

$$dU_i^\gamma = \partial_t U_i^\gamma dt + \partial_y U_i^\gamma dY_t + \frac{1}{2} \partial_{yy} U_i^\gamma d\langle Y_t, Y_t \rangle \quad (3.11)$$

By (3.9):

$$d\langle Y_t, Y_t \rangle = \gamma^2 Y_t^2 \left(\sigma + e^{rt} Y_t^{-\frac{1}{\gamma}} \right)^2 dt. \quad (3.12)$$

Therefore (3.11) becomes

$$\begin{aligned} dU_i^\gamma &= \left[\partial_t U_i^\gamma + \gamma Y_t \partial_y U_i^\gamma \left(r + \frac{\gamma-1}{2} (\sigma + e^{rt} Y_t^{-\frac{1}{\gamma}})^2 \right) \right. \\ &\quad \left. + \frac{1}{2} \gamma^2 Y_t^2 \left(\sigma + e^{rt} Y_t^{-\frac{1}{\gamma}} \right)^2 \partial_{yy} U_i^\gamma \right] dt \end{aligned} \quad (3.13)$$

$$+ \gamma Y_t \partial_y U_i^\gamma \left(\sigma + e^{rt} Y_t^{-\frac{1}{\gamma}} \right) dW_t \quad (3.14)$$

Since the process $(E[Y_t | \mathcal{F}_s])_{s \in [t_{i-1}, t_i]}$ is a martingale, then by the martingale representation theorem 2.2.1 the term in dt of the above equation must vanish. This leads to the PDE

$$\begin{cases} \partial_t U_i^\gamma + \gamma y \partial_y U_i^\gamma \left(r + \frac{\gamma-1}{2} (\sigma + e^{rt} y^{-\frac{1}{\gamma}})^2 \right) + \frac{1}{2} \gamma^2 y^2 \left(\sigma + e^{rt} y^{-\frac{1}{\gamma}} \right)^2 \partial_{yy} U_i^\gamma = 0, \\ U_i^\gamma(t_i, y) = y, \end{cases}$$

which gives the PDE (3.10). □

Now, the third term of (3.6), $E_0^Q \left[\frac{S_{t_i}}{S_{t_{i-1}}} \right]$, can be found.

Corollary 3.2.2. $E_{t_{i-1}}[S_{t_i}] = S_{t_{i-1}} \cdot e^{r(t_i - t_{i-1})}$ and

$$E_0^Q \left[\frac{S_{t_i}}{S_{t_{i-1}}} \right] = e^{r(t_i - t_{i-1})}. \quad (3.15)$$

Proof. Applying Proposition 3.2.1 for $\gamma = 1$ then $Y_t = S_t$ and $E_{t_{i-1}}[S_{t_i}]$ satisfies the PDE

$$\begin{cases} \partial_t U_i + ry \partial_y U_i + \frac{1}{2}(\sigma y + \beta e^{rt})^2 \partial_{yy} U_i = 0 \\ U_i(y, t_i) = y. \end{cases}$$

Let $\tau = t_i - t$ and assume that the solution of the PDE is of the form $U_i(t, y) = ye^{c(\tau)}$,

then

$$\partial_t U_i = \frac{\partial v_i}{\partial t} = \frac{\partial v_i}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = ye^{c(\tau)} \cdot \frac{dc}{d\tau} \cdot (-1)$$

$$\partial_y U_i = e^{c(\tau)} \quad \text{and} \quad \partial_{yy} U_i = 0.$$

Substituting the above PDE to get the ODE

$$e^{c(\tau)} \frac{dc(\tau)}{d\tau} - re^{c(\tau)} = 0$$

subject to the initial condition $c(0) = 0$. This gives $dc(\tau) = rd\tau$ and $c(\tau) = r\tau$. Thus

$U_i(t, y) = ye^{r\tau}$. Therefore

$$\begin{aligned}
 E_0^Q \left[\frac{S_{t_i}}{S_{t_{i-1}}} \right] &= E_0^Q \left[\frac{S_{t_i}}{S_{t_{i-1}}} \right] = E_0^Q \left[\frac{1}{S_{t_{i-1}}} E_{t_{i-1}}^Q [S_{t_i}] \right] \\
 &= E_0^Q \left[\frac{1}{S_{t_{i-1}}} U_i(t_{i-1}, y) \right] \\
 &= E_0^Q \left[\frac{1}{S_{t_{i-1}}} S_{t_{i-1}} \cdot e^{r(t_i - t_{i-1})} \right] \\
 &= e^{r\Delta t}.
 \end{aligned}$$

The proof is complete. □

The second term in (3.6) can't be computed using the PDE in Proposition 3.2.1., however it can be reduced to the computation of conditional expectation in S_t^{-1} .

Proposition 3.2.2. Let

$$C = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} e^{2r\Delta t} - \frac{2\beta}{\sigma} e^{r(3t_i - 2t_{i-1})}$$

and

$$D = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} \cdot \frac{\beta}{2\sigma} \cdot e^{2rt_i}.$$

Then

$$E_0^Q \left[\left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \right] = CE_0^Q \left[\frac{1}{S_{t_{i-1}}} \right] + DE_0^Q \left[\frac{1}{S_{t_{i-1}}^2} \right]. \quad (3.16)$$

Proof. Using (3.6):

$$E_0^Q \left[\frac{S_{t_i}^2}{S_{t_{i-1}}^2} \right] = E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} E_{t_{i-1}}^Q [S_{t_i}^2] \right].$$

Since

$$S_{t_i} = \left(S_0 + \frac{\beta}{\sigma} \right) e^{(r - \frac{\sigma}{2})t_i + \sigma W_{t_i}} - \frac{\beta}{\sigma} e^{rt_i} = \alpha \xi_{t_i} - \frac{\beta}{\sigma} e^{rt_i},$$

where

$$\alpha = S_0 + \frac{\beta}{\sigma} \quad \text{and} \quad \xi_{t_i} = e^{(r - \frac{\sigma}{2})t_i + \sigma W_{t_i}}.$$

Then

$$S_{t_i}^2 = \alpha^2 \xi_{t_i}^2 + \frac{\beta^2}{\sigma^2} e^{2rt_i} - 2 \frac{\beta}{\sigma} \alpha \xi_{t_i} e^{rt_i}.$$

Moreover,

$$E_{t_{i-1}}^Q[S_{t_i}^2] = E_{t_{i-1}}^Q[\alpha^2 \xi_{t_i}^2] + \frac{\beta^2}{\sigma^2} e^{2rt_i} - 2 \frac{\beta}{\sigma} e^{rt_i} E_{t_{i-1}}^Q[\alpha \xi_{t_i}].$$

The calculation of the two conditional expectations in the above equality had been investigated. The second expectation can be easily computed using (3.15). In fact:

$$E_{t_{i-1}}^Q[\alpha \xi_{t_i}] = E_{t_{i-1}}^Q[S_{t_i} + \frac{\beta}{\sigma} e^{rt_i}] = E_{t_{i-1}}^Q[S_{t_i}] + \frac{\beta}{\sigma} e^{rt_i} = S_{t_{i-1}} e^{r\Delta t} + \frac{\beta}{\sigma} e^{rt_i}.$$

To find the first conditional expectation $E_{t_{i-1}}^Q[\alpha^2 \xi_{t_i}^2]$ first notice that

$$\begin{aligned} \alpha^2 \xi_{t_i}^2 &= \alpha^2 \cdot e^{2(r - \frac{\sigma}{2})t_i + 2\sigma W_{t_i}} = \alpha^2 e^{(r_2 - \frac{\sigma_2}{2})t_i + \sigma_2 W_{t_i}} \\ &= \left(\frac{\alpha^2}{S_0 + \frac{\beta}{2\sigma}} \right) \left[\left(S_0 + \frac{\beta}{\sigma_2} \right) e^{(r_2 - \frac{\sigma_2}{2})t_i + \sigma_2 W_{t_i}} + \gamma_2(t_i) \right] - \frac{\alpha^2}{S_0 + \frac{\beta}{2\sigma}} \cdot \gamma_2(t_i). \end{aligned}$$

Then

$$E_{t_{i-1}}^Q[\alpha^2 \xi_i^2] = \frac{\alpha^2}{S_0 + \frac{\beta}{2\sigma}} \left[E_{t_{i-1}}^Q[S_{2t_i}] - \gamma_2(t_i) \right].$$

Where

$$S_{2t_i} = S_{t_i}(r_2, \sigma_2) \text{ and } \gamma_2(t_i) = \frac{-\beta}{2\sigma} e^{2rt_i}.$$

Then,

$$E_{t_{i-1}}^Q[\alpha^2 \xi_i^2] = \frac{\alpha^2}{S_0 + \frac{\beta}{2\sigma}} \left(S_{t_{i-1}} e^{2r\Delta t} + \frac{\beta}{2\sigma} e^{2rt_i} \right),$$

and

$$\begin{aligned} E_{t_{i-1}}^Q[S_{t_i}^2] &= AS_{t_{i-1}} + B + \gamma^2(t_i) + 2\gamma(t_i)[S_{t_{i-1}} e^{r\Delta t} - \gamma(t_i)] \\ &= [A + 2\gamma(t_i) e^{r\Delta t}] S_{t_{i-1}} + B - \gamma^2(t_i), \end{aligned}$$

where $A = \frac{\alpha^2}{S_0 + \frac{\beta}{2\sigma}} e^{2r\Delta t}$ and $B = \frac{\beta}{2\sigma} e^{2rt_i} \cdot \frac{\alpha^2}{S_0 + \frac{\beta}{2\sigma}}$. Therefore:

$$E_{t_{i-1}}^Q[S_{t_i}^2] = CS_{t_{i-1}} + D,$$

where $C = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} e^{2r\Delta t} - \frac{2\beta}{\sigma} e^{r(3t_i - 2t_{i-1})}$ and $D = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} \cdot \frac{\beta}{2\sigma} \cdot e^{2rt_i}$. Finally:

$$\begin{aligned} E_0^Q \left[\left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \right] &= E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} (CS_{t_{i-1}} + D) \right] \\ &= CE_0^Q \left[\frac{1}{S_{t_{i-1}}} \right] + DE_0^Q \left[\frac{1}{S_{t_{i-1}}^2} \right]. \end{aligned}$$

This ends the proof. □

The next proposition summarizes the previous results and provides a formula for K_{var} in high volatile model.

Proposition 3.2.3. *Consider a discretely-sampled variance swap where the underlying asset price is governed by SDE:*

$$dS_t = rS_t dt + (\sigma S_t + \beta e^{rt}) dW_t, \quad S_0 > 0. \quad (3.17)$$

Let the delivery strike value is denoted by:

$$K_{var} = 100^2 * \frac{AF}{N} \left[N + \sum_{i=1}^N \left(CE_0^Q \left[\frac{1}{S_{t_{i-1}}} \right] + DE_0^Q \left[\frac{1}{S_{t_{i-1}}^2} \right] \right) - 2e^{r\Delta t_i} \right]. \quad (3.18)$$

Where there are N observations and AF is representing the annualized factor.

Proof. It is a direct application of corollary 3.1.1, corollary 3.2.2 and proposition 3.2.2. □

Remark: In the previous proposition, the variance swap delivery price is provided with two expectations, $E_0^Q \left[\frac{1}{S_{t_{i-1}}} \right]$ and $E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} \right]$, that are not straight forward. An investigation of computing these two expectations can be a subject of future work.

Chapter 4: Variance Swap in High Volatility Regime Switching Model

In this chapter, the framework of Chapter 3 will be extended to regime-switching by incorporating a Markov-modulated version of the high volatility model. The considered hybrid model has parameters that switch according to a continuous-time Markov chain process. More precisely, a numerical method will be utilized to simulate the price of discrete variance swap for an augmented volatility model. In the next section, the model will be presented. The second section describes the numerical methods. Illustrations and results of comparison with the Heston and the model studied in Chapter 3 are provided in section 3.

4.1 High volatility regime switching model

In this section, calculations of the price of the variance swap in a regime switching model with high volatility will be treated. This model is an extension of the model considered in Chapter 3.

4.1.1 Two-state Continuous-Time Markov Chain (CTMC)

The work will be on a continuous Markov process with two states only. First, a brief introduction to CTMC as presented in [13] will be given. A Two-state Continuous-Time Markov Chain (CTMC) $(X_t)_{0 \leq t \leq T}$ is a continuous Markov process with

1. State space $S = \{0, 1\}$
2. Transition semi-group:

$p(t) = [p_{i,j}(t)]_{i,j \in S}$, where $p_{i,j}(t) = p[x_{t+s} = j | x_s = i]$. In the case of $S = \{0, 1\}$, therefore

$$p(t) = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{bmatrix}$$

3. Markov property: (a) $p(x_t = j | x_s = i_n, x_{s-1} = i_{n-1}, \dots, x_{s_0} = i_0) = p(x_t = j | x_s = i_n)$.

where $0 < s_0 < s_1 < \dots < s_{n-1} < s < t$.

(b) $p_{i,j}(t+s) = \sum_{l \in S} p_{i,l}(s) p_{l,j}(t)$.

4. The infinitesimal generator Q of $(X_t)_{t \in \mathbb{R}_+}$ is

$$Q := p'(0) = \lim_{s \rightarrow 0} \frac{p(s) - p(0)}{s}.$$

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \text{ where } \alpha, \beta > 0.$$

The Forward Kolmogorov equation is given by:

$$\begin{aligned} p'(t) &= p(t)Q \\ &= p(t) \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \end{aligned}$$

The solution is given by:

$$p(t) = \frac{1}{\alpha + \beta} \left\{ \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} e^{-t(\alpha + \beta)} \right\}.$$

4.1.2 The main model

Assume that a variance swap with underlying asset price $(S_t)_{t \in [0, T]}$ is governed by the following SDE

$$\begin{cases} dS_t = r(X_t)S_t dt + [\sigma(X_t)S_t + g(t)]dW_t \\ S_0 > 0. \end{cases} \quad (4.1)$$

Here $g(t)$ is a deterministic function that represents the abnormal increase in the volatility. The interest rate and the σ are depending on X_t as follows

$$r(X_t) = \begin{cases} r_1, \text{ if } X_t = 0 \\ r_2, \text{ if } X_t = 1 \end{cases} \quad (4.2)$$

and

$$\sigma(X_t) = \begin{cases} \sigma_1, \text{ if } X_t = 0 \\ \sigma_2, \text{ if } X_t = 1, \end{cases} \quad (4.3)$$

respectively.

4.2 Numerical simulations for variance swap value

The aim is to provide numerical simulation for the underlying asset price as well as for the fair variance delivery price K_{var} under these settings. This model generalizes the model in chapter three by adding the regime switching to the parameter r and σ .

4.2.1 SDEs Simulation

A variance reduction procedure is utilized to simulate the main model. This is to shorten the number of trials to estimate the fair variance delivery price K_{var} , and to re-

duce the computation time. The literature contains many methods to examine variance reduction procedures among others: Antithetic Variable Technique, Control Variate Technique, Importance Sampling, Stratified Sampling, Moment Matching and Quasi-Random Sequences. The reader can refer to [7] for details. To simulate trajectories for the underlying asset price, a discretization of the SDE (4.1) will be shown. This can be done as follow

$$\begin{cases} \Delta S_{t_n} = r(X_{t_n})S_{t_n}\delta t + [\sigma(X_{t_n})S_{t_n} + g(t_n)]\Delta W_{t_n} \\ S_0 > 0 \text{ is given.} \end{cases} \quad (4.4)$$

That is

$$S_{t_{n+1}} - S_{t_n} = r(X_{t_n})S_{t_n}\delta t + [\sigma(X_{t_n})S_{t_n} + g(t_n)]\Delta W_{t_n} \quad (4.5)$$

$$S_{t_{n+1}} = S_{t_n} + r(X_{t_n})S_{t_n}\delta t + [\sigma(X_{t_n})S_{t_n} + g(t_n)]\Delta W_{t_n} \quad (4.6)$$

The Antithetic Variable Technique will be used. This technique calculate two values of the K_{var} . First, simulate a random variable RV_n , the first value, in usual way; the second value $-RV_n$ is calculated by changing the sign of all the random samples from standard normal distributions. Since the delivery variance value is a function of the asset price S_t , and the asset price S_t is a function of a Brownian motion which is normally distributed, N random variables RV_n can be simulated and then calculate $-RV_n$, so a total of $2N$ random variables will be obtained with only N trials. First calculate from RV_n a first estimation of K_{var} , let's call it \hat{K}_{var1} and from $-RV_n$ a second estimation of K_{var} , let's call it \hat{K}_{var2} , the estimated value of the fair delivery variance swap is calculated as the average of \hat{K}_{var1} and \hat{K}_{var2} i.e.

$$\bar{K}_{var} = \frac{\hat{K}_{var1} + \hat{K}_{var2}}{2}. \quad (4.7)$$

"This works well because when one value is above the true value, the other tends to be below, and vice versa" [7]. Figure 4.1 represents the above technique.

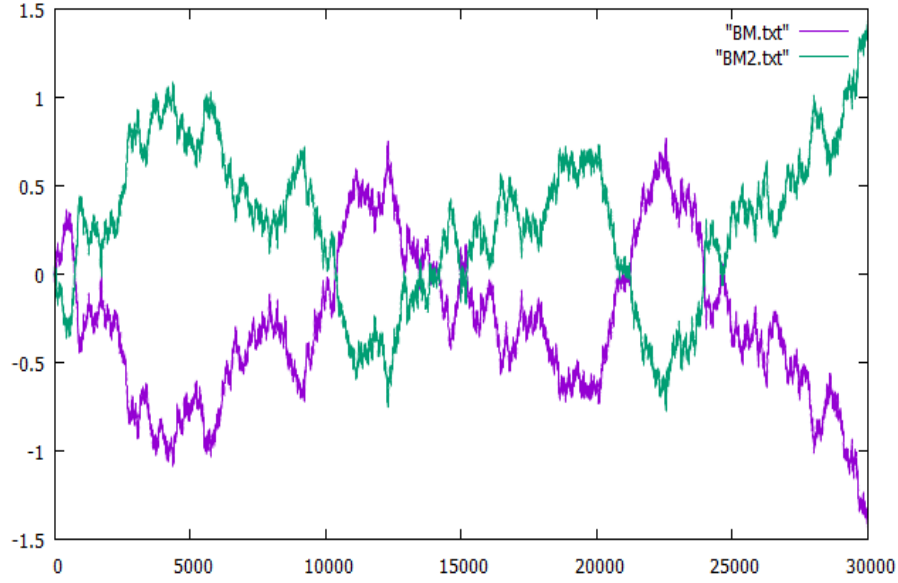


Figure 4.1: A trajectory of Brownian motion BM_n verses $-BM_n$

4.2.2 Simulation of Continuous-Time Markov Chain (CTMC)

The simulation of the main model requires first the simulation of a two state Markov process $(X_t)_{0 \leq t \leq T}$ with $X_t = 0$ or $X_t = 1$. These states represent the economy by either good economy ($X_t = 0$) or poor economy ($X_t = 1$). Let the time period be $t \in [0, T]$. Always assume that the process starts with good economy $X_t = 0$. The process will continue at this state until time $\tau \in [0, T]$, then it will change to the other state $X_t = 1$. The example below presents a simulation of X_t .

Example 4.2.1. A trajectory of X_t could be as follow

$$X_t = \begin{cases} 0, & 0 \leq t \leq \tau_1 \\ 1, & \tau_1 \leq t \leq \tau_2 \\ 0, & \tau_2 \leq t \leq \tau_3 \\ 1, & \tau_3 \leq t \leq \tau_4 \\ 0, & \tau_4 \leq t \leq T. \end{cases}$$

Figure 4.2 is representing the above case:

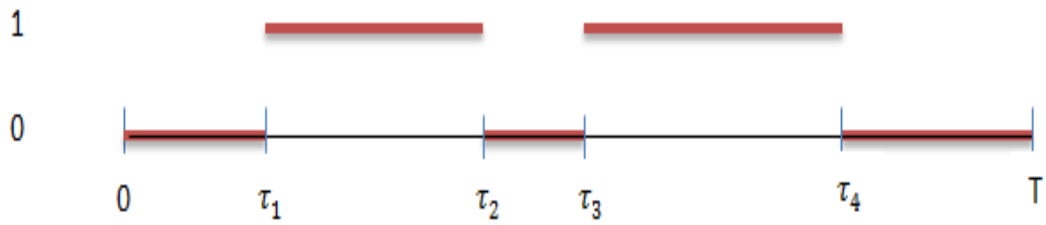


Figure 4.2: A simulation of CTMC for two states

Figure 4.3 below represents how the underlying asset price changes as the states of Markov process change.

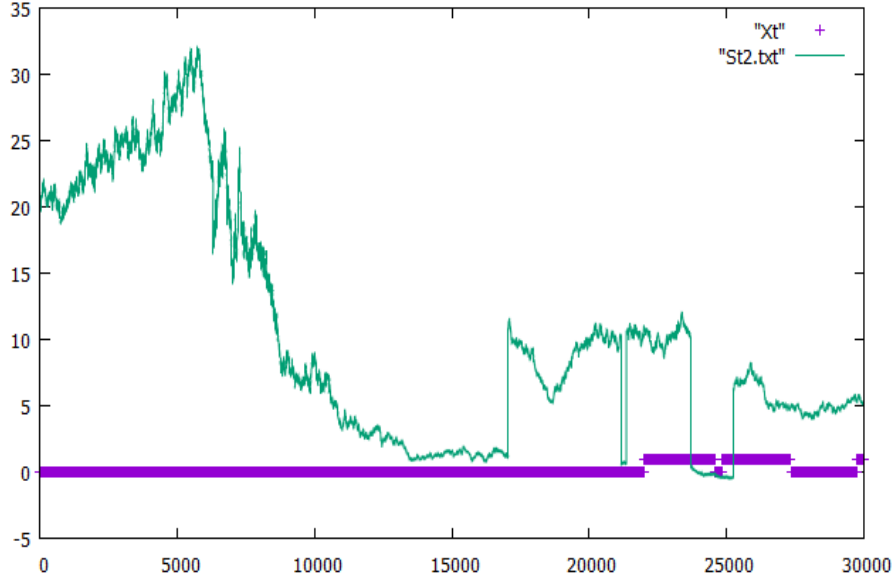


Figure 4.3: The effect of Markov Process on underlying asset price

In the appendix the C++ language code is provided.

4.2.3 Monte Carlo method to price the variance swaps

In the simulation, it has been taken equally-spaced discrete N samples and annualized factor

$$AF = \frac{1}{\Delta t} = \frac{N}{T}. \quad (4.8)$$

The realized volatility is given by

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=1}^N \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 * 100^2. \quad (4.9)$$

S_{t_i} is the i -th closing price observation of the underlying asset. As mentioned in Chapter 3, the fair variance delivery price is $K_{var} = E_0^Q[\sigma_R^2]$. The simulation of the fair

delivery variance swap value is based on the following formula:

$$\begin{aligned}
K_{var} &= \frac{AF}{N} E_0 \left[\sum_{n=0}^N \left(\frac{S_{t_{n+1}} - S_{t_n}}{S_{t_n}} \right)^2 \right] \\
&= \frac{AF}{N} \left[\sum_{n=0}^N E_0 \left(\frac{S_{t_{n+1}} - S_{t_n}}{S_{t_n}} \right)^2 \right] \\
&= \frac{AF}{N} \sum_{n=0}^N E \left[\frac{(S_{t_{n+1}})^2}{S_{t_n}^2} + 1 - 2 \frac{S_{t_{n+1}}}{S_{t_n}} \right] \\
&= \frac{AF}{N} \sum_{n=0}^N \left(1 + E_0 \left[\left(\frac{S_{t_{n+1}}}{S_{t_n}} \right)^2 \right] - 2 E_0 \left[\frac{S_{t_{n+1}}}{S_{t_n}} \right] \right).
\end{aligned}$$

So the aim is to simulate N closing prices $S_{t_n}, n = 1, \dots, N$ then the two expectations in the previous formula. Here S_t is following the SDE (3.8). For seek of comparison, a simulation with the Heston model will be done where the underlying asset price is given by the SDE

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{B}_t^S \\ dv_t = \kappa^* (\theta^* - v_t) dt + \sigma_v \sqrt{v_t} d\tilde{B}_t^V. \end{cases} \quad (4.10)$$

Where $\kappa^* = \kappa + \lambda$ and $\theta^* = \frac{\kappa\theta}{\kappa + \lambda}$ are the risk-neutral parameters and the new parameter λ is the premium of volatility risk. As mentioned before, all the simulations code were performed in the language C++ and the program is given in the appendices. The Figures 4.4 - 4.8 provide trajectories of the underlying asset.

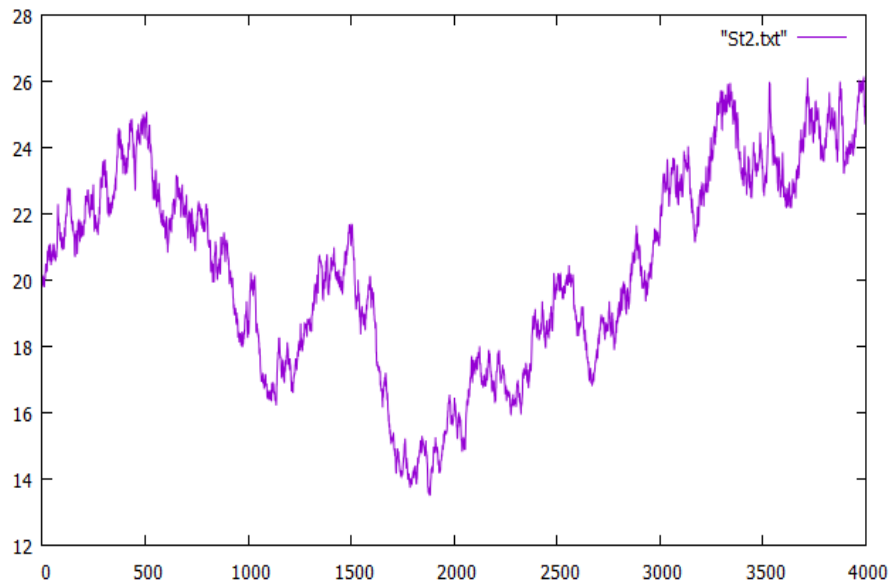


Figure 4.4: The underlying asset price with 9 sampled prices

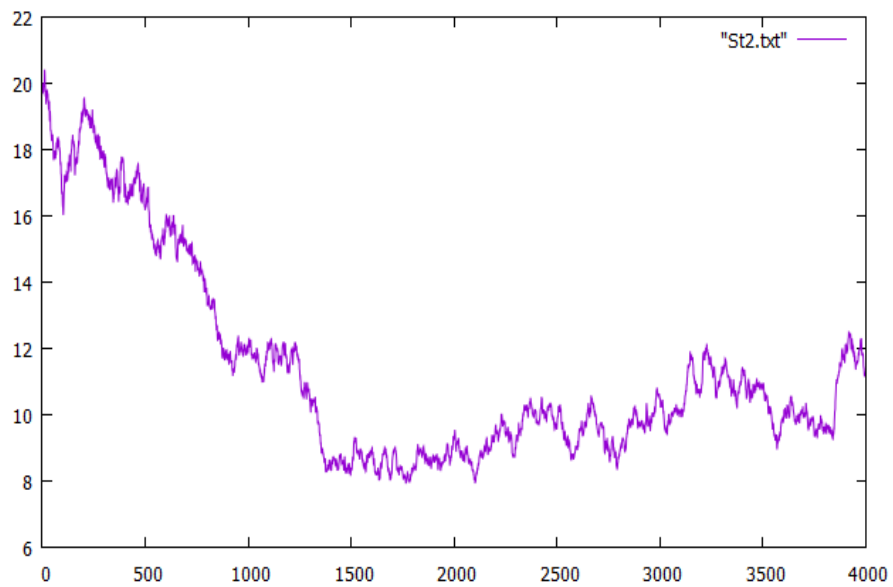


Figure 4.5: The underlying asset price where $\kappa = 3$

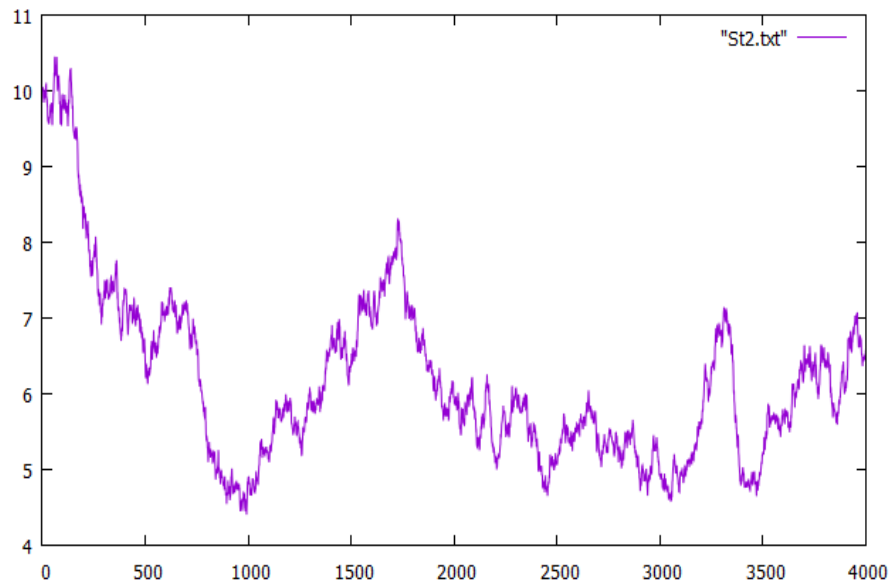


Figure 4.6: The underlying asset price where the initial price = 10

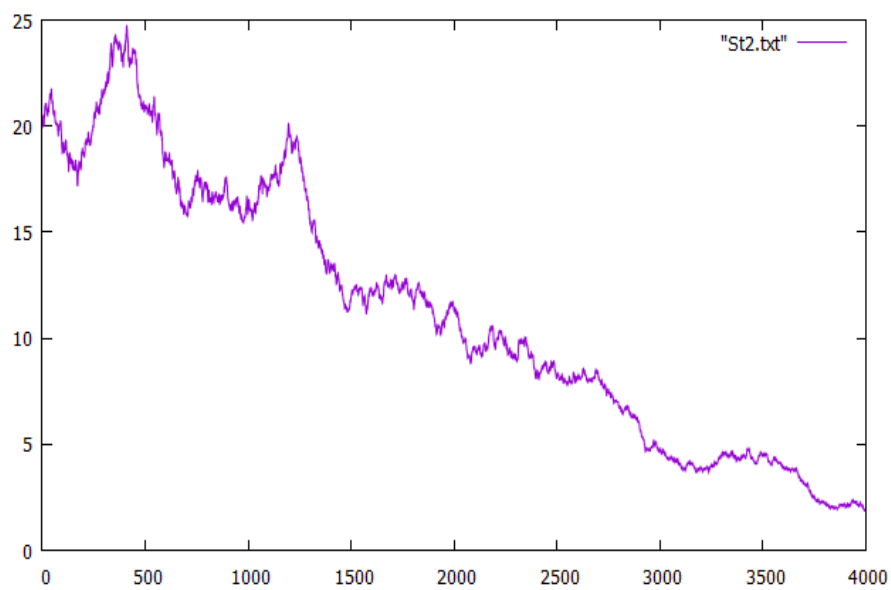


Figure 4.7: The underlying asset price where $\sigma = 0.5$

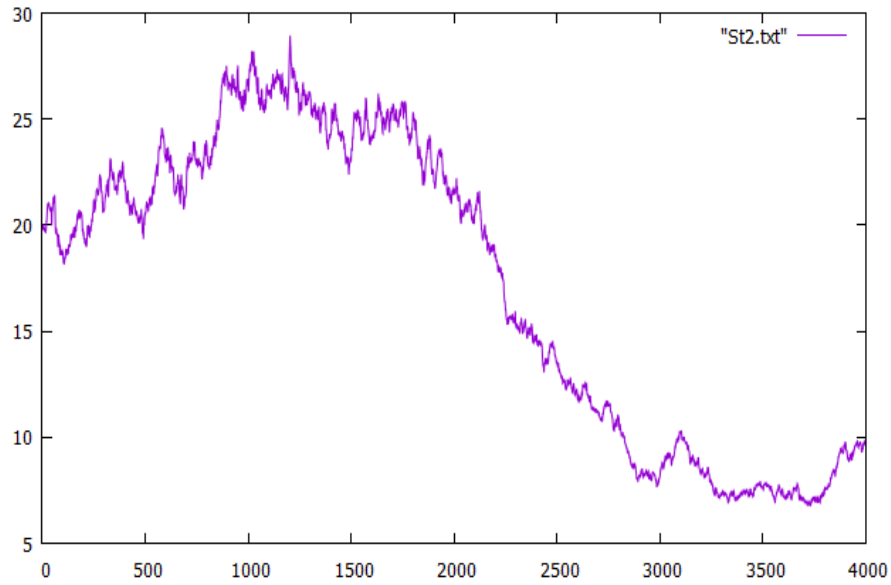


Figure 4.8: The underlying asset price where $\theta = 5$

The Figures 4.9 - 4.13 provide the trajectory of the underlying asset in Heston model.

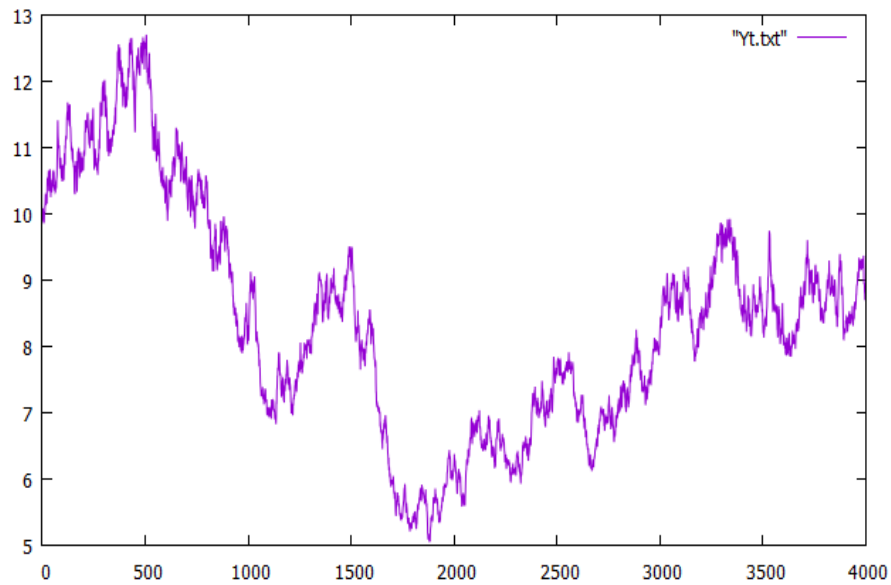


Figure 4.9: The price in Heston model with 9 sampled prices

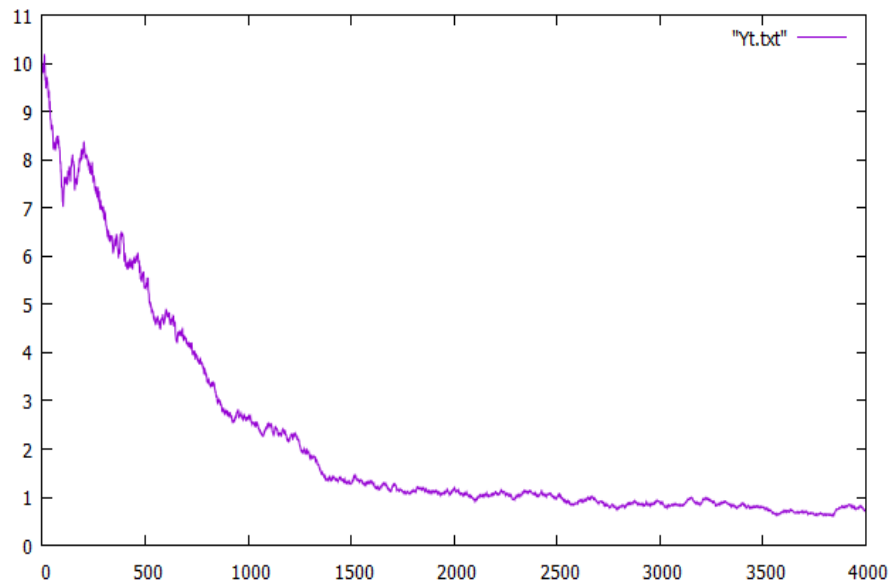


Figure 4.10: The price in Heston model where $\kappa = 3$

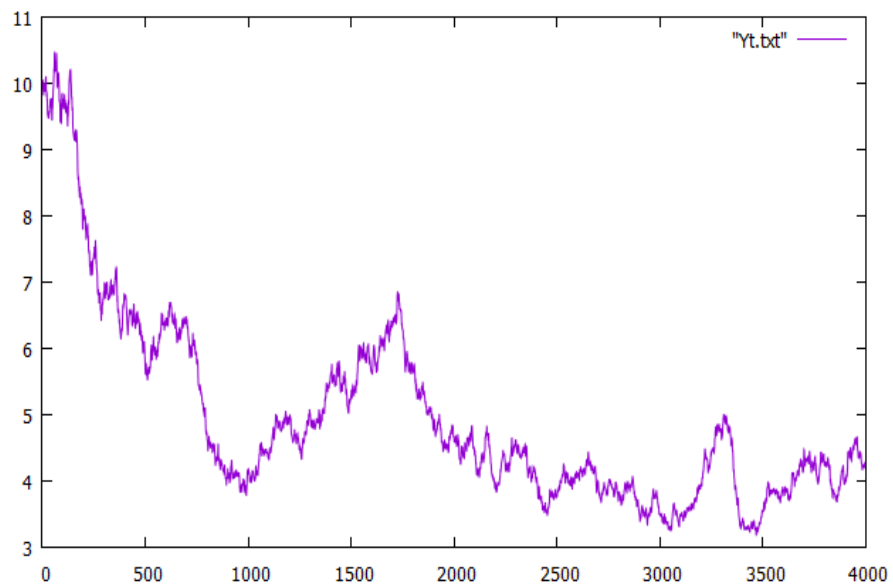


Figure 4.11: The price in Heston model with initial price = 10

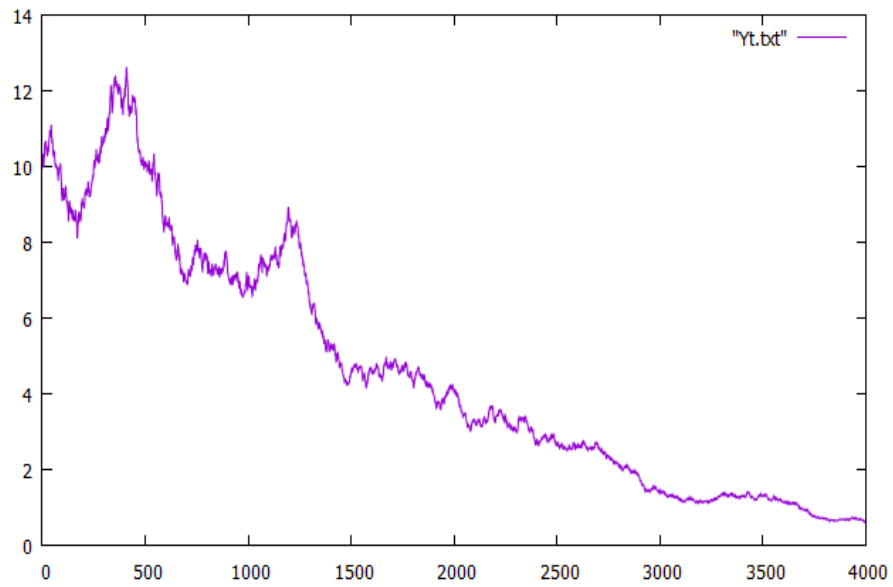


Figure 4.12: The price in Heston model where $\sigma = 0.5$

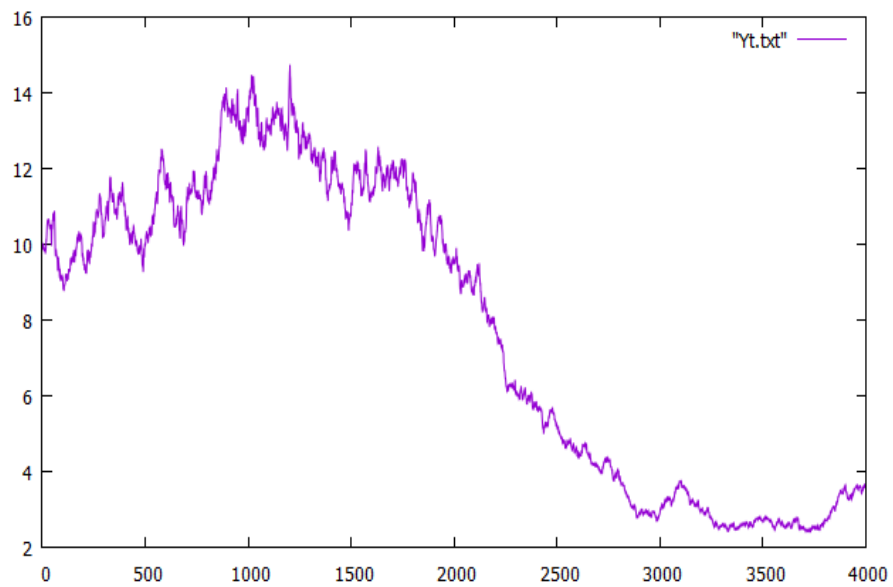


Figure 4.13: The price in Heston model where $\theta = 5$

The Figures 4.14 - 4.18 show the price trajectory of the variance swap.

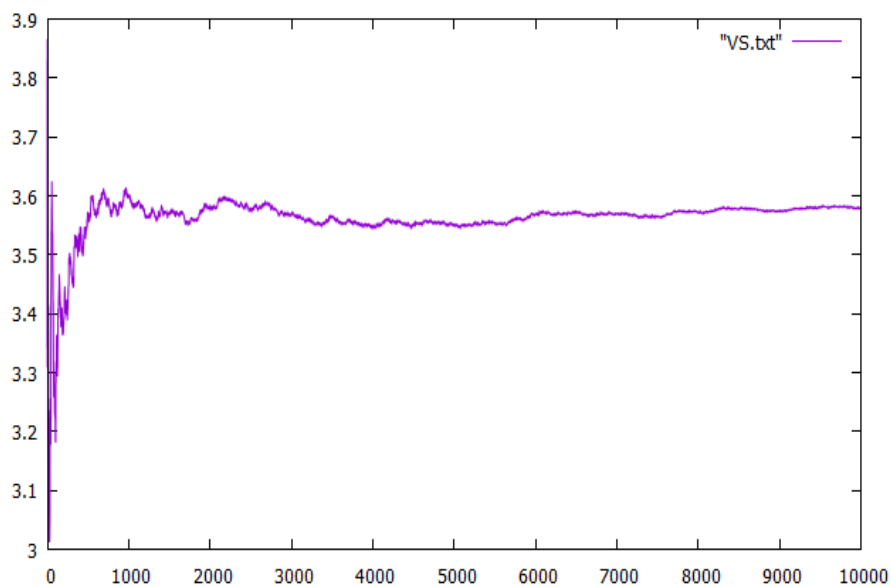


Figure 4.14: The price of variance swap with 9 sampled prices

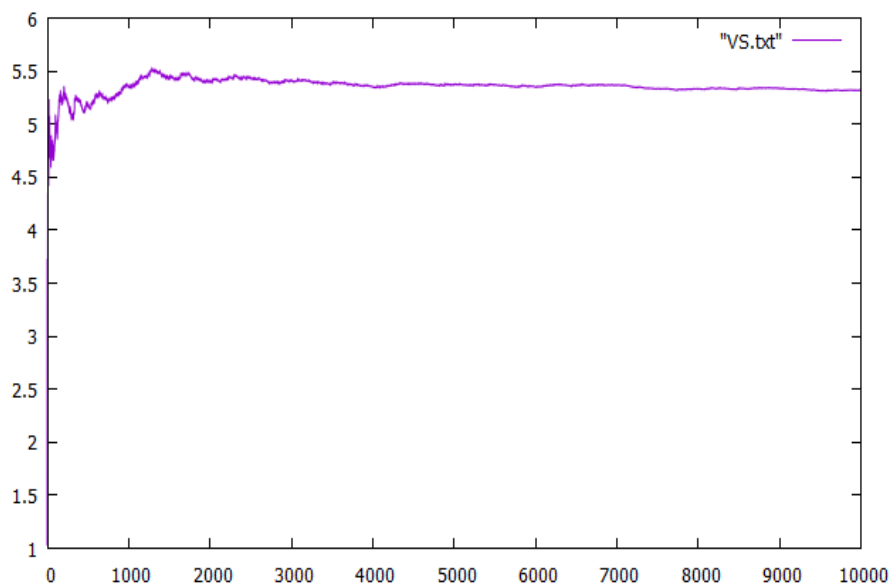


Figure 4.15: The price of variance swap where $\kappa = 3$

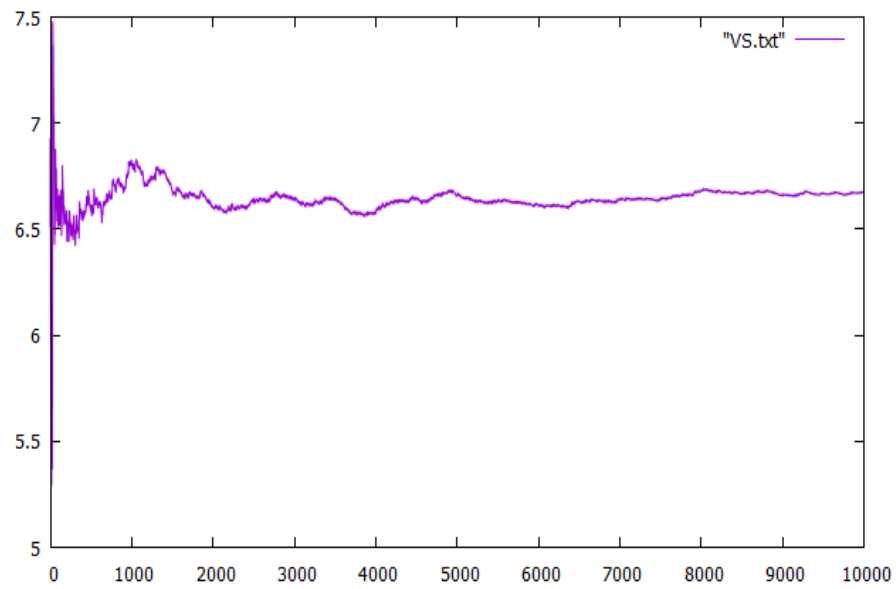


Figure 4.16: The price of variance swap where the initial price = 10

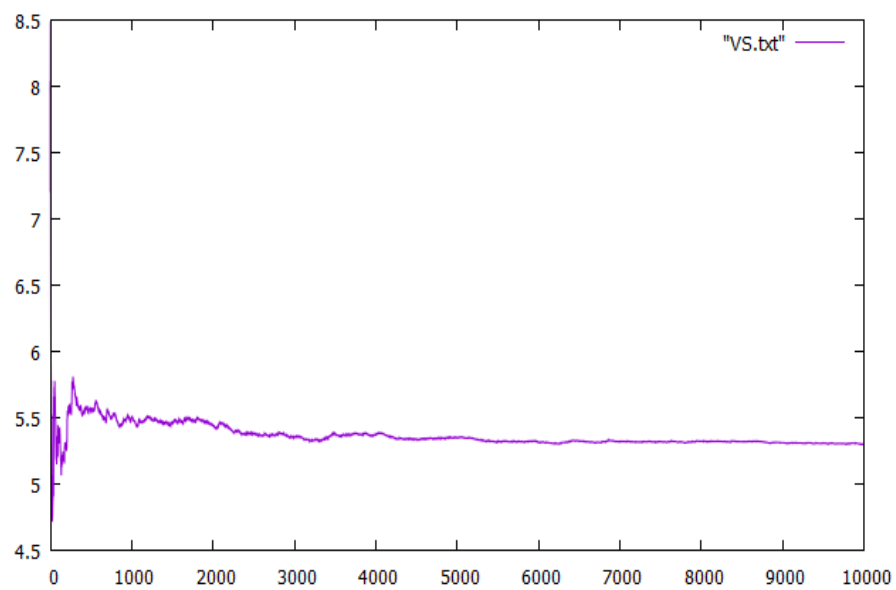


Figure 4.17: The price of variance swap where $\sigma = 0.5$

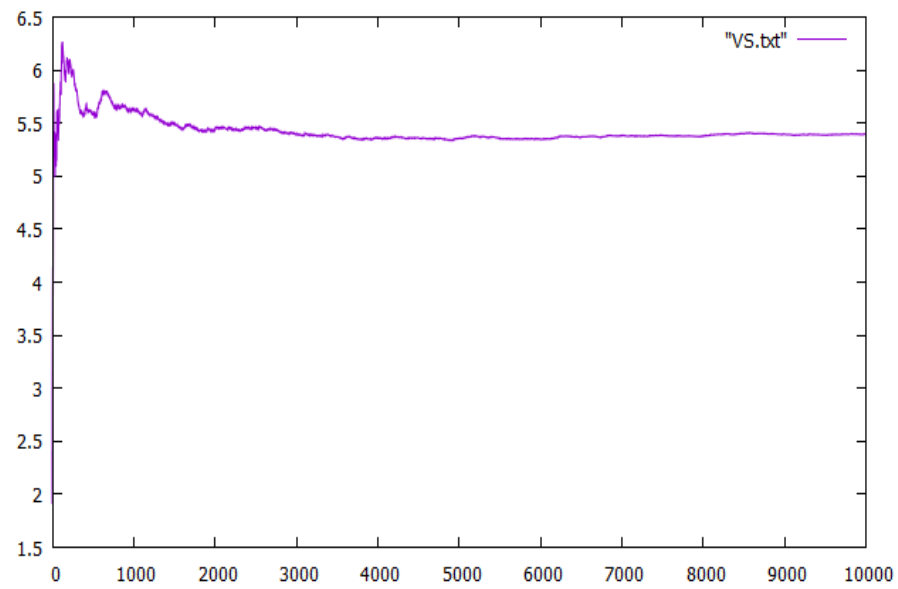


Figure 4.18: The price of variance swap where $\theta = 5$

Chapter 5: Conclusion

In this thesis two underlying asset price models have been employed for the pricing of variance swaps in the discretely sampled case. In first model the underlying asset is assumed to be highly fluctuated. The considered model is important since it accounts for situations where the asset is suffering from high variation in its price. Variance swaps have not been explored in such circumstance to the best knowledge. The price of a discretely-sampled variance swaps has been calculated as the sum of two quantities: first quantity computed theoretically. The second quantity can be a subject of future work.

The second model treated in this thesis aims at generalizing the high volatility model by allowing the underlying asset price to be dependent of a continuous time Markov chain. This model can be seen as high volatile model with regime switching. In second model, theoretical results are very challenging and not sure to be obtained, numerical techniques had been employed to obtain simulation for the fair delivery variance swap prices. The two models treated in this thesis provide some stylized facts that are observed regularly in the markets such as an increase in the volatility, leverage effect, regime switching for instance from bad to good economy and vice-versa. Thus, their importance. As a future work, a closed form solution for the fair delivery price of a variance swap in the case of high volatility with and without regime switching to be investigated.

References

- [1] Albanese, C. & Campolieti, G. (2006). *Advanced Derivatives Pricing and Risk Management*. San Diego: Elsevier.
- [2] Broadie, M. & Jain, A. (2008). The effect of jumps and discrete sampling on volatility and variance swaps. *International Journal of Theoretical and Applied Finance*, 11(8), 761-797.
- [3] Bufngton, J. & Elliott, R. (2002). American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5(5), 497-514.
- [4] Demeter, K., Derman, E., Kamal, M. & Zou, J. (1999). *More than you ever wanted to know about volatility swaps*. New York: Goldman Sachs Quantitative Strategies Research Notes.
- [5] El-Khatib, Y. & Hatemi-J, A. (2017). Option valuation and hedging in markets with a crunch. *Review of Economic Studies*, 44(5), 801-815.
- [6] Heston, S. (1993). A closed-form solution for option pricing with stochastic volatility with application to bond and currency options. *Review of Financial Studies*, 6(2), 327-343.
- [7] Hull, J. (2008). *Options, Futures And Other Derivatives* (Seventh ed.). University of Toronto: Pearson Prentice Hall.
- [8] Javaheri, A., Wilmott, P. & Haug, E. (2004). GARCH and volatility swaps. *Journal of Quantitative Finance*, 4(5), 589-595.
- [9] Little, T. & Pant, V. (2001). A finite-difference method for the valuation of variance swaps. *Journal of Computational Finance*, 5(1), 81-101.
- [10] Oksendal, B. (2000). *Stochastic Differential Equations: An Introduction with Applications, Corrected printing* (Fifth ed.). New York: Springer-Verlag Heidelberg.

- [11] Oksendal, B. & Sulem, A. (2009). *Applied Stochastic Control of Jump Diffusions* (Third ed.). New York: Springer-Verlag Berlin Heidelberg.
- [12] Papanicolaou, A. (2015). *Introduction to Stochastic Differential Equations (SDEs) for Finance*. New York: preprint, arXiv:1504.05309, 2015 - arxiv.org.
- [13] Privault, N. (2018). *Understanding Markov Chains Examples and Applications* (second ed.). Singapore: Springer Singapore.
- [14] Rota, G. M.I.T. (2013). A random variable is neither random nor variable [e-slides]. Retrieved March 3, 2020, from <https://ethz.ch/content/dam/ethz/special-interest/mavt/dynamic-systems-n-control/idsc-dam/Lectures/Stochastic-Systems/Probability.pdf>.
- [15] Rujivan, S. & Zhu, S. (2012). A simplified analytical approach for pricing discretely-sampled variance swaps with stochastic volatility. *Applied Mathematics Letters*, 25(11), 1644–1650.
- [16] Zhu, S. & Lian, G. (2011). A closed-form exact solution for pricing variance swaps with stochastic volatility. *Mathematical Finance*, 21(2), 233–256.
- [17] Zhu, S.P., Badran, A. & Lu, X. (2012). A new exact solution for pricing European options in a two-state regime-switching economy. *Computers & Mathematics with Applications*, 64(8), 2744–2755.

Appendices

Appendix 1: Probability Space

A probability space has three components (Ω, F, P) , respectively the sample space, event space, and probability function. The set Ω is the sample space, which is the set of the outcomes of an experiment. An event F is a subset of Ω . Finally, a probability function P assigns a number (probability) to each event in F . It is a function mapping $F \rightarrow [0, 1]$ satisfying: [14]

1. $P(A) \geq 0$, for all $A \in F$.
2. $P(\Omega) = 1$
3. Countable additivity: If $A_1, A_2, \dots \in F$ are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset, \forall i \neq j$), then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

A very important thing is that the event F must satisfy the σ - algebra conditions. Therefore, let's define a σ - algebra. A collection F of subsets of Ω is called a σ - algebra on Ω if:

1. $\Omega \in F$, and $\emptyset \in F$.
2. If $A \in F$, then $\Omega - A = A^c \in F$: The complementary subset of A is also in Ω .
3. For all $A_i \in F : \cup_{i=1}^{\infty} A_i \in F$.

The smallest σ - algebra is $\{\emptyset, \Omega\}$ while the largest σ - algebra is Ω .

Remark: Given any family B of subset of Ω , denote by:

$$\sigma(B) := \cap \{C : C, \sigma - \text{algebra of } \Omega, B \subset C\}$$

the smallest σ - field of Ω containing B, $\sigma(B)$ is called the σ - field generated by B. When B is a collection of all open sets of a topological space Ω , $\sigma(B)$ is called the Borel σ - algebra on Ω and the elements of $\sigma(B)$ are called Borel sets.

Remark: If $X : \Omega \rightarrow \mathbb{R}^n$ is a function, then the σ -algebra generated by X is the smallest

σ -algebra on Ω containing all the sets of the form

$$\{X^{-1}(U) : U \subset \mathbb{R}^n, \text{open}\}$$

Definition 6.1.1 (Negligeable set)

(i) Given a probability space (Ω, F, P) , A subset Ω is said to be P -null or negligeable if $P(A) = 0$.

(ii) A property is said to be true almost surely (a.s) if the set on which this property is not true is negligeable.

Definition 6.1.2 (Measurability and random variable)

(i) Let (Ω, F, P) and (Ω', F', P') be two probability spaces. A function $X : \Omega \rightarrow \Omega'$ is said to be F -measurable if and only if

$$X^{-1}(U) := \{\omega \in \Omega : X(\omega) \in U\} \in F, \quad \forall U \in F'.$$

(ii) A random variable X is a function $X : \Omega \rightarrow \Omega'$ F -measurable.

(iii) If $\Omega' = \mathbb{R}$, then X is called a real random variable.

(iv) If $\Omega' = \mathbb{R}^n$, $n > 1$ then X is called a vector random variable.

All the coming notations will denote that (Ω, F, P) as a probability space and X as a random variable, $X : \Omega \rightarrow \mathbb{R}^n$.

Remark: Every random variable induces a probability measure on \mathbb{R}^n denoted μ_X and define by $\mu_X(B) := P(X^{-1}(B))$, $\forall B$ open set of \mathbb{R}^n . μ_X is called the distribution function of X .

Definition 6.1.3 (Expected value)

(i) If X is a random variable such that $\int_{\Omega} \|X(\omega)\| dP(\omega) < \infty$ almost surely, the quantity

$$E(X) := \int_{\Omega} X(\omega) dP(\omega) = \int_{\mathbb{R}^n} x d\mu_X(x)$$

is called the expected value of X , where $||\cdot||$ denote the euclidean norm on \mathbb{R}^n .

(ii) In general, if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, is measurable and $\int_{\Omega} ||f(X(\omega))|| dP(\omega) < \infty$ almost surely, then the quantity $E(f(X))$ define by

$$E(f(X)) := \int_{\Omega} f(X(\omega)) dP(\omega) = \int_{\mathbb{R}^n} f(x) d\mu_X(x)$$

is called expected value of $f(X)$.

Definition 6.1.4 (Independent random variables)

Let (Ω, F, P) be a probability space.

1. Two elements A and B of F are independent if

$$P(A \cap B) = P(A) \cap P(B).$$

2. Two random variables X_1 and X_2 of (Ω, F, P) are independent if for every choice of different borel sets B_1 and B_2 the following holds:

$$P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1) * P(X_2 \in B_2).$$

Proposition 6.1.1 Two random variables X_1 and X_2 are independent if and only if for any measurable positive functions f_1 and f_2 , the following equality holds [11]

$$E(f_1(X_1)f_2(X_2)) = E(f_1(X_1))E(f_2(X_2)).$$

Definition 6.1.5 (Conditional probability)

For any event A such that $P(A) > 0$, the conditional probability on A is the probability measure define by:

$$P(B|A) := \frac{P(A \cap B)}{P(A)}, \quad \forall B \in F.$$

Definition 6.1.6 Let X be a random variable such that $\int_{\Omega} |X(\omega)| dP(\omega) < \infty$ almost surely. Let \mathbf{G} a sub σ -algebra of F . The conditional expectation of X relative to the σ -algebra \mathbf{G} is a random variable denoted by $E(X/\mathbf{G})$ satisfying

1. $E(X/\mathbf{G})$ is \mathbf{G} -measurable.
2. $\int_G E(X/\mathbf{G}) dP = \int_G X dP, \forall G \in \mathbf{G}$. Note that $E(X/\mathbf{G})$ is called the projection of X upon \mathbf{G} .

Proposition 6.1.2 (i) $E(E(X/\mathbf{G})) = E(X)$.

(ii) If X is \mathbf{G} -measurable, then $E(X/\mathbf{G}) = X$.

(iii) $E((X+Y)/\mathbf{G}) = E(X/\mathbf{G}) + E(Y/\mathbf{G})$.

(iv) If $\mathbf{G} \subset \mathbf{G}'$, then $E(X/\mathbf{G}') = E(E(X/\mathbf{G})/\mathbf{G}')$.

(v) If $\sigma(X)$ and \mathbf{G} are independent, then $E(X/\mathbf{G}) = E(X)$.

(vi) If $X \leq Y$ a.s, then $E(X/\mathbf{G}) \leq E(Y/\mathbf{G})$.

(vii) If X is \mathbf{G} measurable, then $E(XY/\mathbf{G}) = XE(Y/\mathbf{G})$.

Definition 6.1.7 (Convergence of random variables)

Let $p \in [1, \infty)$, let $L^p(\Omega, \mathbb{R}^n)$ be the equivalence class of measurable functions $X : \Omega \rightarrow \mathbb{R}^n, F_t$ - measurable such that

$$(\|X\|)_{L^p(\Omega, \mathbb{R}^n)}^p := E(\|X\|^p) = \int_{\Omega} \|X(\omega)\|^p dP(\omega) < +\infty.$$

Let $(X_n) \subset L^p(\Omega, \mathbb{R}^n)$ be a sequence of random variables and $X \in L^p(\omega, \mathbb{R}^n)$ a random variable. Let

$$N := \{\omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)\}$$

1. (X_n) converges to X almost surely if N^c is negligible.

2. (X_n) converges in probability to X if

$$\forall \varepsilon > 0 \quad \lim_{n \rightarrow \infty} P(\|X_n - X\| > \varepsilon) = 0.$$

3. (X_n) converges in L^p to X if

$$\lim_{n \rightarrow +\infty} E(|X_n - X|^p) = 0.$$

Appendix 2: Project Simulation Code

Remark: The using software, G++, does not have the normal distribution while it has the uniform distribution. Therefore, the following technique will be used in order to convert the uniform distribution to a normal distribution.

Box–Muller transform: Suppose U_1 and U_2 are independent samples chosen from the uniform distribution on the unit interval $(0, 1)$. Let

$$Z_0 = R \cos(\theta) = \sqrt{-2 \ln U_1} \cos(2\pi U_2)$$

$$Z_1 = R \sin(\theta) = \sqrt{-2 \ln U_1} \sin(2\pi U_2)$$

Then Z_0 and Z_1 are independent random variables with a standard normal distribution.

// Simulation_Project1.cpp : This file contains the 'main' function. Program execution begins and ends there.

```
#include "pch.h"
```

```
#include "Simh1.h"
```

```
#include <iostream>
```

```
#include <fstream>
```

```
#include<stdio.h>
```

```
#include<math.h>
```

```
#include <stdlib.h>
```

```
#include <ctype.h>
```

```
#include <time.h>
```

```
#include "CTMC.h"
```

```
using namespace std;
```

```

int main()

{

int i = 0;

int j = 0;

int m = 0;

int TNN = 0;

double* normal;

double* normal1;

double* normal2;

double* Bt1;

double* Bt2;

//process Ornstien

double* Yt;

double Y0 = 10.;

//process lognormal

double* St;

double S0 = 20.;

double* SS0;

double* SSt1;

double* SSt2;

double* SSt3;

double EEvar = 0.; //expected value from variance swap
formula sampled

double KK = 0.; // strike value of the variance swap

```

```

double AF = 100.;

int Nsim = 1000; // number of simulation

int N0 = 7; //mber of sampled prices for the varianve
swap formula

TNN = (int(Nbdisc - 1) / (N0 - 1));

int* Xt; // Markov chain process

ofstream fXt;

ofstream fBt;

ofstream fBt2;

ofstream fYt;

ofstream fSt1;

ofstream fSt2;

ofstream fVS; // value of the variance swap

srand(time(NULL));

Yt = (double *)malloc(Nbdisc * sizeof(double));

for (i = 0; i < Nbdisc; i++)

Yt[i] = 0.;

St = (double *)malloc(Nbdisc * sizeof(double));

for (i = 0; i < Nbdisc; i++)

St[i] = 0.;

normal = (double *)malloc(2 * Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal[j] = 0.;

normal1 = (double *)malloc(Nbdisc * sizeof(double));

```

```

for (j = 0; j < Nbdisc; j++)

normal1[j] = 0.;

normal2 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal2[j] = 0;

Bt1 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

Bt1[j] = 0.;

normal = Normal();

normal1 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal1[j] = normal[j];

normal2 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal2[j] = normal[Nbdisc + j];

Bt1 = Brownien(normal1);

//std::cout << Bt1[100];

fBt.open("BM.txt");

for (i = 0; i < Nbdisc-1; i++) {

fBt << Bt1[i];

fBt << "\n";

}

fBt << Bt1[Nbdisc - 1];

fBt.close();

```

```

fBt2.open("BM2.txt");

Bt2 = Brownien(normal2);

for (i = 0; i < Nbdisc-1; i++) {

fBt2 << Bt2[i];

fBt2 << "\n";

}

fBt2 << Bt2[Nbdisc - 1];

fBt2.close();

fYt.open("Yt.txt");

Yt[0] = Y0;

for (i = 0; i < Nbdisc -1; i++) {

Yt[i + 1] = Yt[i] + kappa0 * (theta0 - Yt[i])

* deltat + gamma0 *

Yt[i]*(Bt1[i + 1] - Bt1[i]);

fYt << Yt[i];

fYt << "\n";

}

fYt.close();

fSt1.open("St1.txt");

St[0] = S0;

for (i = 0; i < Nbdisc - 1; i++) {

St[i + 1] = St[i] + r0 * St[i] * deltat

+ sqrt(Yt[i]) * St[i] *

(Bt1[i + 1] - Bt1[i]);

```

```

fSt1 << St[i];

fSt1 << "\n";

}

fSt1.close();

fSt2.open("St2.txt");

Xt = (int *)malloc(Nbdisc * sizeof(int));

for (int i = 0; i < Nbdisc; i++)

Xt[i] = 0;

Xt = CMarkovChain();

St[0] = S0;

for (i = 1; i < Nbdisc; i++) {

if(Xt[i]==0)

St[i] = (St[0] + 1/sigma0) *

exp((r0 - sigma0 * sigma0 / 2)*

deltat*i + sigma0 * Bt1[i]) -

exp(r0*deltat*i) / sigma0;

else

St[i] = (St[0] + 1 / sigma1) *

exp((r1 - sigma1 * sigma1 / 2)*

deltat*i + sigma1 * Bt1[i]) -

exp(r1*deltat*i) / sigma1;

fSt2 << St[i];

fSt2 << "\n";

}

```

```

fSt2.close();

/* Calculations of variance swap

SS0 = (double *)malloc(Nsim * sizeof(double));

// initialisation

for (i = 0; i < Nsim; i++)

SS0[i] = 0.;

SSt1 = (double *)malloc(Nsim * sizeof(double));

for (i = 0; i < Nsim; i++)

SSt1[i] = 0.;

SSt2 = (double *)malloc(Nsim * sizeof(double));

for (i = 0; i < Nsim; i++)

SSt2[i] = 0.;

SSt3 = (double *)malloc(Nsim * sizeof(double));

for (i = 0; i < Nsim; i++)

SSt3[i] = 0.;

fVS.open("VS.txt");

fXt.open("Xt");

for (int n = 0; n < Nbdisc; n++)

{

fXt << Xt[n];

fXt << "\n";

}

fXt.close();

for (int ll = 0; ll < Nsim - 1; ll++)

```



```

// Nbsim number of simulation of the price process

{

free(normal);

free(normal1);

free(Bt1);

free(Bt2);

free(St);

free(normal2);

free(Xt);

St = (double *)malloc(Nbdisc * sizeof(double));

for (i = 0; i < Nbdisc; i++)

St[i] = 0.;

normal = (double *)malloc(2 * Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal[j] = 0.;

normal1 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal1[j] = 0.;

normal2 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal2[j] = 0;

Bt1 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

Bt1[j] = 0.;

```

```

Bt2 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

Bt2[j] = 0.;

normal = Normal();

normal1 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal1[j] = normal[j];

normal2 = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

normal2[j] = normal[Nbdisc + j];

Xt = (int *)malloc(Nbdisc * sizeof(int));

for (int i = 0; i < Nbdisc; i++)

Xt[i] = 0;

Xt = CMarkovChain();

Bt1 = Brownien(normal1);

Bt2 = Brownien(normal2);

St[0] = S0;

for (i = 1; i < Nbdisc; i++) {

if(Xt[i]==0)

St[i] = (St[0] + 1/sigma0) *

exp((r0 - sigma0 * sigma0 / 2)*

deltat*i + sigma0 * Bt1[i]) -

exp(r0*deltat*i) / sigma0;

else

```

```

St[i] = (St[0] + 1 / sigma1) *
exp((r1 - sigma1 * sigma1 / 2)*
deltat*i + sigma1 * Bt1[i]) -
exp(r1*deltat*i) / sigma1;
//fSt2 << St[i];
//fSt2 << "\n";
}

for (int nn = 1; nn < N0; nn++) {
EEvar = EEvar + pow((St[nn*TNN] - St[(nn - 1)*TNN]) /
St[(nn - 1)*TNN], 2);
}

fVS << (EEvar / (11+1))*AF / N0;
fVS << "\n";
}

EEvar = (1./Nsim)*EEvar;
KK = (AF / N0) * EEvar;

cout << KK;

fVS.close();
}

```

Header File Code

```

#pragma once

#define Pi 3.1415161718

int Nbdisc = 30001; //number of discretization of

```

```

the asset price trajectory

#define T 3. // expiration date in years

double deltat = double(T / Nbdisc);

#include <stdlib.h>

#include<math.h>

double kappa0 = 0.3;

double theta0 = 0.5;

double gamma0 = 0.7;

double r0 = 0.1;

double sigma0 = 0.5;

double kappal = 1.3;

double theta1 = -0.05;

double gamma1 = 0.07;

double r1 = 0.01;

double sigma1 = 1.2;

double kappa(int XX) {

if(XX==0)

return kappa0;

else

return kappal;

}

double theta(int XX) {

if (XX == 0)

return theta0;

```

```

else

return theta1;

}

double sigma(int XX) {

if (XX == 0)

return sigma0;

else

return sigma1;

}

double gamma(int XX) {

if (XX == 0)

return gamma0;

else

return gamma1;

}

double r(int XX) {

if (XX == 0)

return r0;

else

return r1;

}

//Simulation of the Uniform Law U(0,1)

double urand() {

return (double)((double)rand() / (double)RAND_MAX);

```

```

}

//Simulation of the Normal Law U(0,1)

double NormDist(double m, double sigma) {

double unif1;

double unif2;

do

unif1 = (double)urand();

while (unif1 == 0.0 || unif1 == 1.);

do

unif2 = (double)urand();

while (unif2 == 0.0 || unif2 == 1.);

return m + (double)sqrt(sigma)*sqrt(-2.*log(unif2))*

cos(2.*Pi*unif1);

}

double* Normal()

{

int j;

double *Normal;

Normal = (double *)malloc(2 * Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

{

Normal[j] = NormDist(0., 1.);

Normal[Nbdisc + j] = -Normal[j];

}

```

```

return Normal;

}

double* Brownien(double* normal)

{

int j;

double *Wt;

Wt = (double *)malloc(Nbdisc * sizeof(double));

for (j = 0; j < Nbdisc; j++)

Wt[j] = 0.;

for (j = 0; j < Nbdisc - 1; j++)

Wt[j + 1] = Wt[j] + (double)sqrt((double)deltat)*normal[j];

return Wt;

}

```

CTMC with two states

```

// To Generate a CTMC: Continuous Time Markov Chain with
two states: S={0,1}

//----- Simulation of the exponential Law-----

//-----exp(lambda),Exp=1./lambda,
Variance = 1./lambda^2-----

double loi_exp(double lambda) {

double unif;

do

unif = (double)urand();

```

```

while (unif == 1.0);

return ((double)(-1. / lambda)*(double)log(1 - unif));

}

//***** Infinitesimal Generator [-tg1  tg1]

//***** Infinitesimal Generator [tg2  -tg2]

const double tg1 = 1.1;

const double tg2 = 1.4;

//*****Continuous Markov Chain *****

//***Simulation of a couple (Holding time H_i,

StateValue_Up between

H_{i-1} and H_i)

//***Simulation of StateValue_Up between H_{i-1} and

H_i int Regimeswitch(int CS)

{

if (CS == 0)

return 1;

else

return 0;

}

//***Simulation of a couple (Holding time H_i,

StateValue_Up between H_{i-1} and H_i)

int* CMarkovChain()// create the holding time

and the State Value

given the infinitesimal generator, this is

```



```

a specific two-state case

{

int CounterHT = 0;

int CurrentState = 0;

double Rate[2]; //Infinitesimal generator matrix

//double GenratorHT=0.; //Genrator of R.V

~Exp(Rates[i])==Hi

double t = 0.; // time varying: t=t0=0....t=t1=t0+H0...

t=t2=t1+H1....stop    when t>T

int* MMXt; //will return the Xt

MMXt = (int *)malloc(Nbdisc * sizeof(int));

for (int j = 0; j < Nbdisc; j++)

MMXt[j] = 0;

Rate[0] = tg1;

Rate[1] = tg2;

t = loi_exp(Rate[CurrentState]);

if (t < T)

{

while (CounterHT < t*Nbdisc / T)

{

MMXt[CounterHT] = CurrentState;

CounterHT++;

}

}

}

```

```

if (CounterHT < Nbdisc)
{
while (t <= T && CounterHT < Nbdisc)
{
CurrentState = Regimeswitch(CurrentState);
t = t + loi_exp(Rate[CurrentState]);
if (t < T)
{
while (CounterHT < t*Nbdisc / T)
{
MMXt[CounterHT] = CurrentState;
CounterHT++;
} // while
} //if
} //while
}

//cout<<"\n \n \n The counter for"
<<CounterHT << "\n \n \n";

if (CounterHT < Nbdisc)
{
for (int j = CounterHT - 1; j < Nbdisc; j++)
MMXt[j] = CurrentState;
}

return MMXt;}

```